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APPROXIMATE METHODS OF SIMULATING RANDOM NOISE WITH PURE TONE SOURCES

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FOREWORD

The work presented in this report was performed by Mr. C. C. Thiel of the General Technology Corporation, Elgin Illinois, for the Aero-Acoustics Branch, Vehicle Dynamics Division, AF Flight Dynamics Laboratory, Wright-Patterson Air Force Base under Contract Nr. AF 33(657)-10945. This research is part of an effort to simulate complex acoustic waves with an array of harmonic sources. This work was performed under Project Nr. 4437 "High Intensity Sound Environment Simulation" Task Nr. 443705 "Development of Noise Sources." Mr. O. F. Maurer of the Aero-Acoustics Branch, AF Flight Dynamics Laboratory was project engineer. The work was undertaken from May 1963 through June 1964. The group supervisors were Dr. A. C. Eringen and Dr. J. C. Samuels. Contractors report number is General Technology Technical Report Nr. 4-3.

This report has been reviewed and is approved.

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ABSTRACT

The problem of the approximation of a given random signal with modulated pure tone signals is studied from the standpoint of approximation of the power spectrum. The equal energy and moment matching techniques are developed for the selections of the modulation parameters of the pure tone sources. Techniques are developed for the selection of parameters of a switched pure tone source. The power spectrum is also approximated by a rational polynomial scheme, which uses a minimax procedure, developed in the appendices, for selection of pure tone operating parameters. Several examples of each of these techniques are given.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

Illustrations (continued)

INTRODUCTION

In a previous report [9] the theoretical problems associated with the approximating of random acoustical signals with pure tone sirens has been treated. This report deals with the more practical aspects of the engineering approximation of random signals. A number of techniques are developed and applied to a particular signal to illustrate the methods.

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I. POWER SPECTRAL DENSITY DETERMINATION

Given a continuous record of finite length, it is not possible to estimate the autocovariance function $C(\tau)$ for arbitrary τ. Thus in place of

$$C(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)x(t+\tau)dt \tag{1.1}$$

we may calculate

$$C_{oo}(\tau) = \frac{1}{T_n-|\tau|}\int_{-(T_n-|\tau|)/2}^{(T_n-|\tau|)/2} x(t-\frac{\tau}{2})\, x(t+\frac{\tau}{2})dt \tag{1.2}$$

where C_{oo} is known as the apparent autocovariance function. Here T_n is the length of the record which is restricted by $|\tau| \leq T_m < T_n$, T_m being the maximum lag time to be considered. We deal with the problem of the relation of T_m to T_n in section III.

Let $D(\tau)$ be a function defined by

$$D(\tau) = \begin{cases} 1 & |\tau| < T_m \\ 0 & |\tau| \geq T_m \end{cases}$$

Now define

$$C_o(\tau) = D(\tau)\, C_{oo}(\tau)$$

It is clear that $C_o(\tau)$ is calculable for a given record, and also that $C_o(\tau)$ is defined for $|\tau| \geq T_m$, even though $C_{oo}(\tau)$ was not defined in this region.

If we assume that the random process is ergodic then the average of a large number of finite records is equivalent to a single average taken over a single record of infinite length. Thus we can write:

$$\text{ave}\{C_o(\tau)\} = D(\tau)C(\tau)$$

It follows then that this relation will have a well defined Fourier integral of the form

$$\text{ave } \{P_o(f)\} = Q(f) * P(f) = \int_{-\infty}^{\infty} Q(\ell)\, P(\ell - f)\, d\ell$$

where $Q(f)$ is the Fourier transform of $D(\tau)$ and $P(f)$ is the Fourier transform of $C(\tau)$, termed the power spectral density. The symbol "*" denotes the convolution.[1] The Fourier transform $G(f)$ of a function $g(\ell)$ is defined, as usual, by

$$G(f) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\ell)\, e^{i\omega\ell}\, d\ell$$

where $f = 2\pi\omega$. Thus,

$$\text{ave } \{P_o(f)\} = \int_{-\infty}^{\infty} Q(f-\ell) P(\ell)\, d\ell. \tag{1.3}$$

We then identify the average of $P_o(f)$ as a smoothing (average over frequency) of the true power spectral density with a weight function $Q(f-\ell)$.

We now turn our attention to the problem of non-continuous records of finite length. Suppose that we have a function $x(t)$ recorded on a record of finite length with the function $x(t)$ specified only at a number of equally spaced points on the time axis t. Let us also assume that $x(t)$ is given only over equally spaced intervals of difference Δt, i.e.

$$t = 0, \Delta t, 2\Delta t, \ldots n\Delta t$$

then $C(\tau)$ can only be estimated for

$$|\tau| = 0, \Delta t, 2\Delta t, \ldots n\Delta t$$

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[1] If $H(f)$ is the Fourier transform of $h(t)$ and $G(f)$ is the Fourier transform of $g(t)$, then the Fourier transform of $h(t)g(t)$ is given as

$$G(f)*H(f) = \int_{-\infty}^{\infty} G(\ell)H(\ell - f)\, d\ell = \int_{-\infty}^{\infty} G(\ell - f)H(\ell)\, d\ell$$

See [1] p. 183 for a further discussion.

The integral equation

$$C(\tau) = \int_0^\infty 2P_A(f) \cos(2\pi f\tau)df$$
$$|\tau| = q\Delta t,\ q = 0,1,\ldots n, \tag{1.4}$$

if soluble at all, must be satisfied by a function $P_A(f)$ which vanishes for $f > f_n = \frac{1}{2\Delta t}$, even though the original power spectral density extends beyond $|f_n|$. This introduces a new problem, that was not encountered in the case of continuous spectra, namely that of aliasing.[2] Consider the two signals shown in Figure 1.

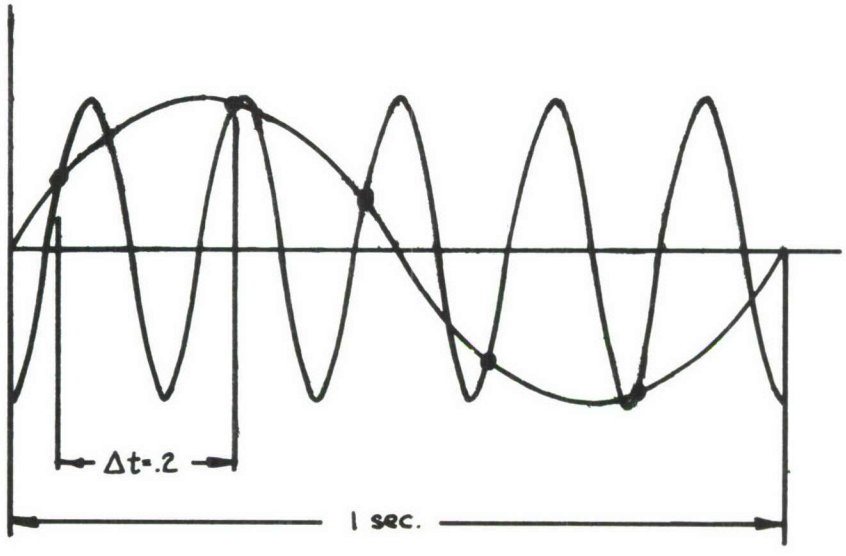

Fig. 1. - Sampling of two sinusodal waves

We note that in terms of the discrete sample $x(t)$, $t = .2q$, $q = 0,1,2,\ldots$, the two waves of different frequency both give the same result. Thus we see that equally spaced time samples of one sine wave could have come from any of many other sine waves. Special care must therefore be exercised in regard to statistical properties of a continuous function $x(t)$. In particular, the relation of the actual power spectral density $P(f)$ and the one

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[2] [2], p. 33

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which we are capable of measuring, $P_A(f)$, represent a critical problem.

To prevent such a difficulty, we will assume that our sampling time is so chosen that the maximum frequency component of $x(t)$ is less than $1/2\Delta t$ i.e.,

$$|f| \leq \frac{1}{|2\Delta t|}$$

This then forces:

$$P(f) = \begin{cases} P_A(f) & |f| \leq (2\Delta t)^{-1} \\ 0 & |f| > (2\Delta t)^{-1} \end{cases} \tag{1.5}$$

It should be remarked that this is seldom the case in actual practice.

One would now ask - "Is there a better sampling scheme than that of equally spaced samples?" If one indeed exists, it appears that it has escaped attention. When the problem of aliasing becomes important, it is usual to follow some method of filtering and/or smoothing the data to make appropriate modifications in the computations. We make no attempt to do this in the present work. References [2], [3] and [4] are of interest in approaching this problem.

II. NUMERICAL DETERMINATION OF POWER SPECTRA

Let us denote the values of the sample function determined at times $t = q\Delta t$, $q = 0,1,2,\ldots n$ as x_q, that is we define

$$x_q = x(q\Delta t) \quad , \quad q = 0,1,\ldots n$$

It is easily seen that the expression analogous to (1.2) is now

$$C_r = C(r\Delta t) = \frac{1}{n-r} \sum_{q=0}^{n-r} x_q x_{q+r} \tag{2.1}$$

where $\Delta\tau = r\Delta t$ and $r=0,1,2,\ldots m \leq n$.

To determine the power spectrum then we take the appropriate analogue to the relation (1.4) and find that

$$P_r = 2\Delta\tau \left\{C_o + 2\sum_{q=1}^{m-1} C_q \cos\frac{qr\pi}{m} + C_m \cos r\pi\right\} \tag{2.2}$$

$$P_r \equiv P\left(\frac{r}{2m\Delta t}\right)$$

$$r = 0,1,2,\ldots,m$$

The factor of 2 appears in eqn. (2.2) due to the fact that we wish to consider only those power spectra symmetric about $\omega = 0$. Thus, given a particular record, we have a numerical method to compute the covariance function and the power spectral density.

If there are several distinct records to be used, as is usually the case, we then define

$$\begin{aligned} \bar{C}_r &= \text{ave } \{C_r\} \\ \bar{P}_r &= \text{ave } \{P_r\} \end{aligned} \qquad (2.3)$$

where the average is taken for each r over each distinct record. By the property of ergodicity, which is tacitly assumed, we are assured that in the limit of increasingly large numbers of records that $\bar{C}_r$ will almost certainly converge to $C(\tau)$ at points of definition of $\bar{C}_r$. For simplicity from here on, we drop the bars on $\bar{C}_r$ and $\bar{P}_r$.

It is also quite frequently desirable to know the probability density for a given random function. The probability density function $P(y)$ is defined as

$$P(y)dy = \text{Prob}\{y \leq x < y + dy\}.$$

Its determination for our discrete sampling then turns out to be merely a counting process.

III. PLANNING FOR THE MEASUREMENTS

Blackman and Tukey [2] have approached the problem of the amount of data required about a particular process in order to make statements concerning the reliability of estimates of power spectra and covariance functions. Because of the astronomical number of calculations involved in estimating a power spectrum for a desired precision, it is important that we compromise between resolution and stability. We shall only reproduce certain parts of their results. The reader will find the details of arriving at these results in section A. 23 → A. 28, B. 23 → B. 24 of [2].

We define T_m, as used in section I, as

$$(\text{resolution in c.p.s.}) = \frac{1}{T_m} \cong R \tag{3.1}$$

where T_m is measured in seconds. "Resolution" is a measure of the concentration of a spectral estimate in frequency units, i.e. a measure of the width of a frequency band in which no attempt is made to further discriminate. We can express the stability associated with a particular estimate in terms of a spread in db, $db = 20 \log_{10} P_1/P_2$, of an interval containing, with prescribed probability, the ratio of true power to estimated power. To this end we define

$$\begin{aligned} k &= 1 + \frac{250}{(80\% \text{ Range In db})} \\ k &= 1 + \frac{400}{(90\% \text{ Range In db})} \\ k &= 1 + \frac{625}{(96\% \text{ Range In db})} \\ k &= 1 + \frac{840}{(98\% \text{ Range In db})} \end{aligned} \tag{3.2}$$

By 90 per cent range, it must be emphasized that we have 9 out of 10 chances of finding each individual estimate within the prescribed db range of its average value. The number of records, P, of length T_n required to yield the particular stability and resolution chosen is then expressible as:

$$P = \frac{\frac{1}{2}(1 + k)}{RT_n - \frac{1}{3}} \tag{3.3}$$

We find that we need

$$m = 3T_n f_{max}$$

lags for the computation of the covariance and $\Delta t = 1/3f_{max}$ as the size of the time intervals. f_{max} is the maximum frequency component present in $x(t)$. Using this, we can compute the required number of data points to be:

$$n = (1.5\,k + P)T_m f_{max} \tag{3.4}$$

We now give several illustrative examples.

Example 1.

Suppose that we are given records of length 1 second having frequency components up to 10^4 c.p.s. and desire a resolution of 50 c.p.s. We wish to know how many samples of this record must be made to have a confidence level of 90 per cent at $\pm$ 2 db. Thus we calculate:

$$T_m = \frac{1}{50} = 2.\times 10^{-2}$$

$$k = 101$$

$$P = 1$$

$$m = 600$$

$$n = 3 \times 10^4$$

we then need only 1 record.

Example 2.

Suppose we are given records of length 2 seconds. We wish to find out how many pieces we will need to have a 98 per cent confidence at $\pm$ 1 db, with maximum frequency components of 10^4 c.p.s. and with 50 c.p.s. resolution.

$$T_m = 2 \times 10^{-2}$$

$$k = 841$$

$$P = 4.2 \sim 4$$

4 records are then required.

IV. AMPLITUDE AND FREQUENCY MODULATED SIGNALS

We have at our disposal K(≤ 25) low frequency and L(≤ 9) high frequency single tone generators (sirens) with which we wish to approximate a given numerical power spectral density. The siren may be either amplitude or frequency modulated within certain restrictive ranges.

Let us find the power spectral densities for each of the three modes of operation of a single siren:

a) Pure tone: $X_k(t) = \sqrt{A_k} \cos \omega_k t$

then $P_k(\omega) = A_k \delta(\omega-\omega_k)$ (4.1)

ω_k is the center frequency of the siren, and A_k its power and $\delta(\omega-\omega_k)$ is the usual Dirac delta function defined as

$$\delta(\omega-\omega_k) = 0 \quad ; \omega \neq \omega_k$$

such that

$$\int_{-\infty}^{\infty} \delta(\omega-\omega_k)d\omega = 1$$

b) Amplitude Modulation

$$x_k(t) = \sqrt{A_k} (1 + \cos \Omega_k t) \cos \omega_k t \qquad (4.2)$$

then

$$P_k(\omega) = \frac{A_k \Delta_k}{4} \{\delta(\omega-\omega_k + \Omega_k) + \delta(\omega-\omega_k-\Omega_k)\} + A_k \delta(\omega-\omega_k)$$

$\sqrt{\Delta_k}$ is called the modulation factor and $\Omega_k/2\pi$ the modulation frequency.

c) Frequency Modulation

$$X_k(t) = \sqrt{A_k} \cos (\omega_k t + \delta_k \sin \Omega_k t) \; ; \qquad (4.3)$$

then

$$P_k(\omega) = A_k \{\sum_{\ell=1}^{\infty} J_\ell^2 [\delta(\omega-\omega_k-\ell\Omega_k) + \delta(\omega-\omega_k + \ell\Omega_k)] + J_o^2 (\delta_k) \delta(\omega-\omega_k)\}$$

δ_k is termed the deviation ratio, and $\Omega_k/2\pi$ the modulation frequency. $J_n(\delta)$ is the Bessel function of order n and argument δ.

Figures 2 and 3 give representative pictures of power spectra of the pure tone and amplitude modulated signals.

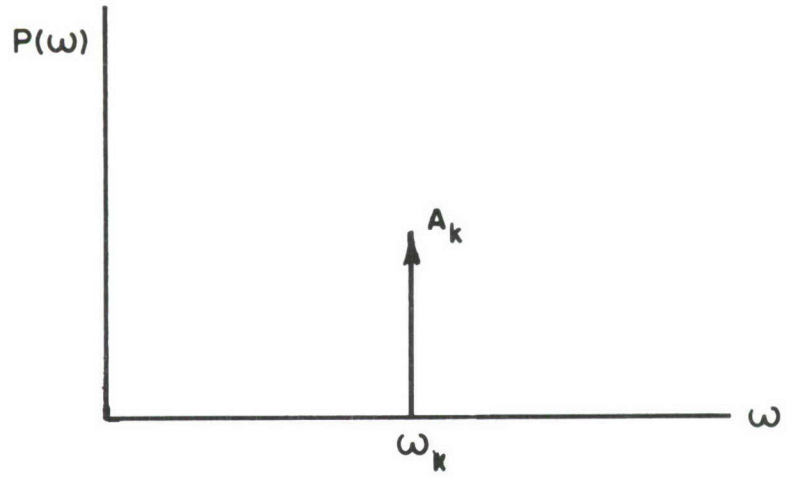

Fig. 2 - Power spectrum of a pure tone signal

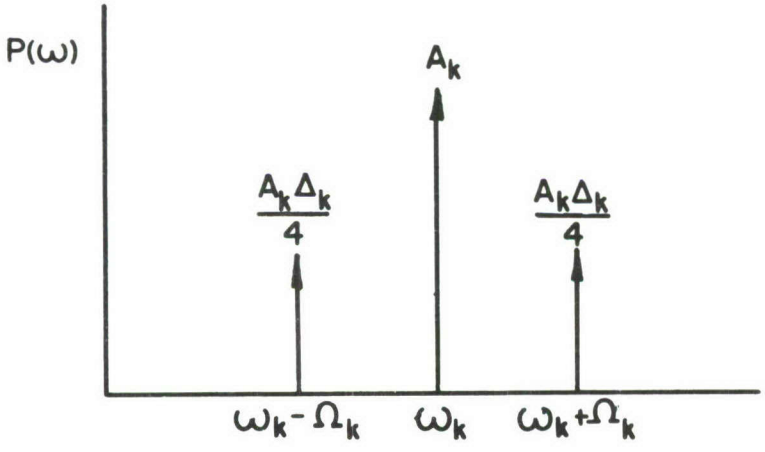

Fig. 3 - Power spectrum of an amplitude modulated signal

The power spectrum of the frequency modulated spectrum is not as simply illustrated. In Figure 4 we illustrate several cases of frequency modulation. A table giving the absolute magnitudes of each frequency component is presented in appendix C.

Suppose we are given a particular power spectral density, S, that is specified numerically at each of the points $\omega = n\Delta\omega$. Let

$$S_n \equiv S(n\Delta\omega) \; ; \; n = 0,1,2,\ldots$$

Our objective is to approximate S by the use of K low frequency sirens, i.e. operating in a frequency range of 0 to $\xi/2\pi$, and L high frequency sirens, operating in a range $\xi/2\pi$ to $\bar{\xi}$, each operating in one of the 3 modes given above by a), b), or c). Each of the three types of modulation above depend upon the specification of a center frequency ω_k. To determine ω_k we make the requirement that each of the K, and similarly the L, sirens have the same total power output. This assumption is made in order to assure that any criteria used for the selection of parameters will lead to realizable power distributions among the several sirens. The total power to be distributed among the K low frequency sirens is given by

$$P_L = \int_0^{\xi} S(\omega)d\omega = \Delta\omega \sum_{q=0}^{r} S_q$$

$$r = \frac{\xi}{\Delta\omega}$$

and similarly for the L high frequency sirens

$$P_H = \int_{\xi}^{\bar{\xi}} S(\omega)d\omega = \Delta\omega \sum_{q=r}^{t} S_q$$

$$t = \frac{\bar{\xi}}{2\pi}$$

We have assumed that $S(\omega)$ is essentially zero for $\omega > \bar{\xi}$. Suppose that Q_L and Q_H are respectively the operating powers of the low and high frequency sirens. Then

$$Q_L = \frac{P_L}{K}$$

$$Q_H = \frac{P_H}{L} \qquad (4.4)$$

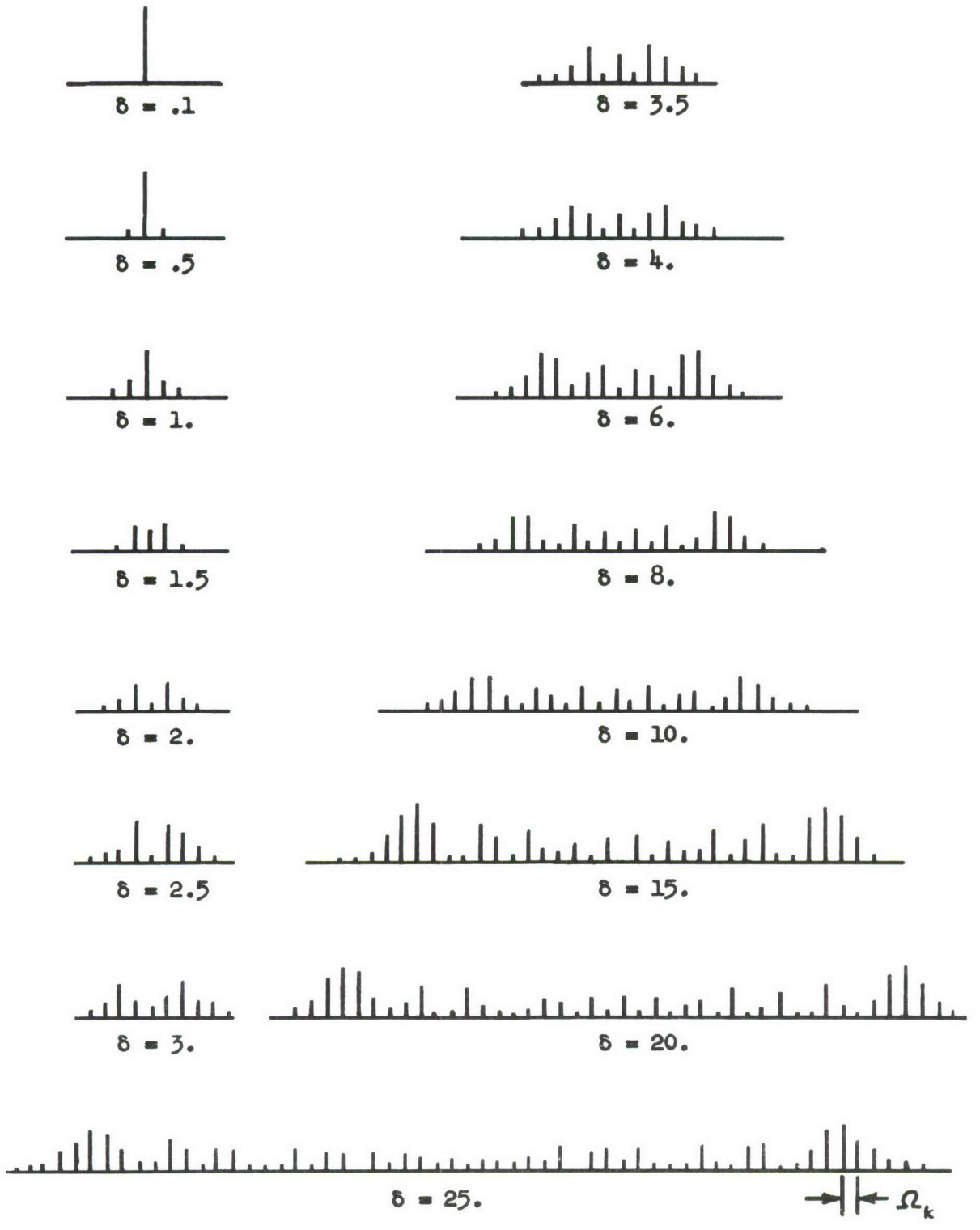

Fig. 4 - Power spectrum of a frequency modulated signal for various δ

If Q_H is above the allowable physical output of a high frequency siren, then it is necessary to scale S in such a manner that Q_H is reduced to this limiting value. We similarly scale S if Q_L is outside the allowable operating range of the low frequency sirens. The interval $(0,\xi)$ now needs to be broken up into K equal energy intervals. Let $(\varphi_k, \varphi_{k+1})$ be the frequency interval in which the k^{th} sirens operates. Then φ_i is determined by:

$$Q_L = \int_{\varphi_i}^{\varphi_{i+1}} S(\omega)d\omega \; ; \; i = 1,2,\ldots, K$$

subject to the conditions that

$$\varphi_1 = 0$$

$$\varphi_{K+1} = \xi$$

Similarly we section the high frequency range by requiring

$$Q_H = \int_{\varphi_i}^{\varphi_{i+1}} S(\omega)d\omega \quad ; \quad i = K+1,\ldots, L$$

where

$$\varphi_{K+1} = \xi$$

$$\varphi_{K+L+1} = \bar{\xi}$$

Thus we have partitioned the power spectrum into parts (see Figure 5).

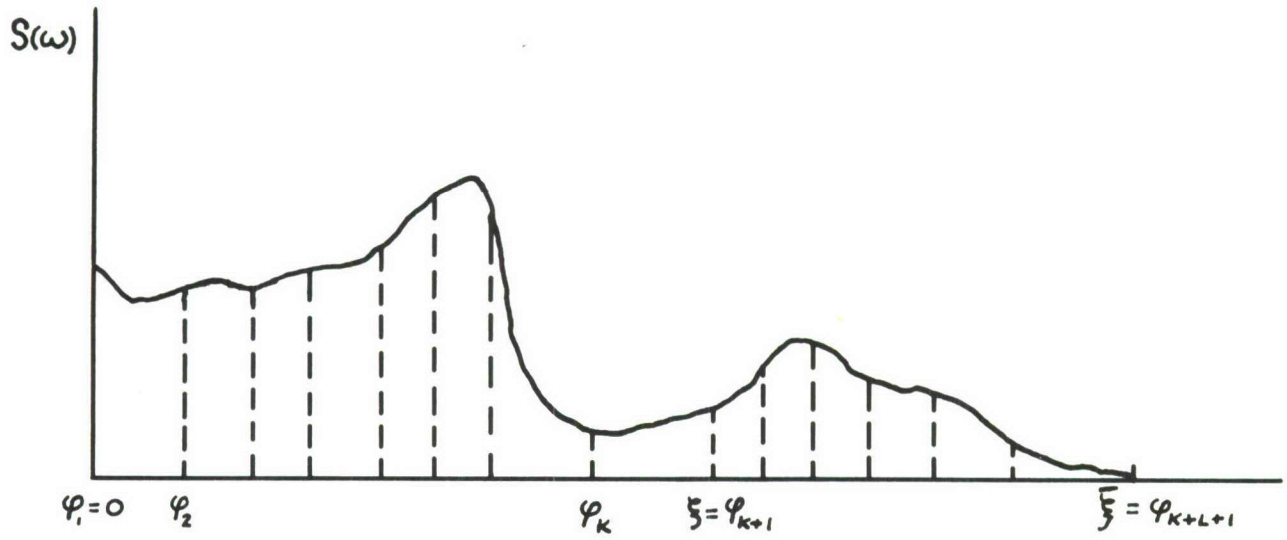

Fig. 5 - Energy partitioning

Let us now examine one frequency band, as illustrated in Figure 6.

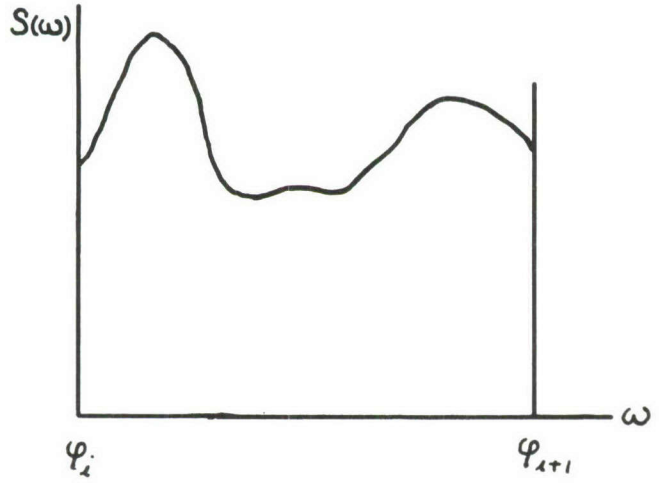

Fig. 6 - A single frequency band

We now make the restriction that only one siren is allowed to operate in each band.

We quote a powerful theorem of Fourier analysis:

Theorem:[2] If $\{T_n\}$ is a complete set of orthonormal functions on the interval $[a,b]$, then the sequence of functions $X_m = \sum_{I}^{M} C_n T_n$ tends uniformly to $X(\omega)$ as $n \to \infty$ at each point $\omega \in [a,b]$ of continuity of X. C_n is the Fourier coefficient defined by

$$C_n = \int_a^b T_n(t)X(t)dt$$

The Tschebyscheff polynomials form such a system of functions. Then n^{th} Tchebyscheff polynomial is a polynomial of degree n in the free variable, that is

$$T_n(z) = \sum_{m=0}^{n} d_m z^m$$

Using this fact we then can claim, by the properties of uniform convergence, that if we force all the moments Γ_n of a function S_2 to be the same as all the moments of another function S_1 then S_2 and S_1 are the same functions. Guided by this we intend to use the moment matching between the actual power spectrum and the assumed power spectrum as the criteria for selecting all the parameters that need to be determined. Generally we will not have enough free parameters in order to force all of the moments to be equivalent. In this case, we will arbitrarily choose the lowest moments needed to specify the parameters.

Let us apply this scheme to the amplitude modulated signal given by eqn. (4.2). The pure tone signal will then be the same except Δ_1 is set identically zero. Writing the balance of the zeroth, first, second, and fourth moments gives us:

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[2] See any book on advanced mathematics such as [1]. It turns out that we must restrict x(t) such that $t^m x(t)$ is square integrable for each m. However, we need not worry about this, since our x(t) will be bounded and continuous on a finite interval.

$$\int_{\varphi_i}^{\varphi_{i+1}} S(\omega)d\omega = A_i + \frac{\Delta_i A_i}{2} = Q_i = \begin{cases} P_L/K & i \leq K \\ P_H/L & i > K \end{cases}$$

$$\int_{\varphi_i}^{\varphi_{i+1}} \omega S(\omega)d\omega = (A_i + \frac{A_i \Delta_i}{2}) \omega_i$$

$$\int_{\varphi_i}^{\varphi_{i+1}} (\omega-\omega_i)^2 S(\omega)d\omega = \frac{A_i\Delta_i\Omega_i^2}{2}$$

$$\int_{\varphi_i}^{\varphi_{i+1}} (\omega-\omega_i)^3 S(\omega)d\omega = \frac{A_i\Delta_i\Omega_i^4}{2}$$

We can then write the parameters A_i, Δ_i, Ω_i, ω_i as

$$\begin{aligned} \omega_i &= \frac{A}{Q_i} \\ \Omega_i &= \frac{\bar{Y}}{\bar{X}} \\ \Delta_i &= \frac{-2\bar{X}^2}{\bar{X}^2 - Q_i\bar{Y}} \\ A_i &= \frac{Q_i}{1 + \frac{\Delta_i}{2}} \end{aligned} \tag{4.5}$$

where:

$$\begin{aligned} \bar{X} &= B - 2\omega_i A + \omega_i^2 Q_i \\ \bar{Y} &= D - 4\omega_i C + 6\omega_i^2 B - 4\omega_i^3 A + \omega_i^4 Q_i \\ A &= (\Delta\omega)^2 \sum_{n=N_1}^{N_2} nS_n \\ B &= (\Delta\omega)^3 \sum_{n=N_1}^{N_2} n^2 S_n \end{aligned} \tag{4.6}$$

$$C = (\Delta\omega)^4 \sum_{n=N_1}^{N_2} n^3 S_n$$

$$D = (\Delta\omega)^5 \sum_{n=N_1}^{N_2} n^4 S_n$$

$$N_1 = \frac{\varphi_i}{\Delta\omega} \quad ; \quad N_2 = \frac{\varphi_{i+1}}{\Delta\omega}$$

Thus we have determined the parameters of the i^{th} siren given that we wish to approximate the power spectrum with an amplitude modulated signal.

In the case of frequency modulation, the situation is not quite as straightforward. We again have four parameters to select, namely, A_i, ω_i, Δ_i, δ_i . First notice that the only place that δ_i occurs is in the argument of the Bessel function J_n. It is fairly obvious that there will be no simple method of choosing δ_i by the solution of the moment matching equations. Thus we must choose δ_i somewhat arbitrarily. First notice that equation $(4.5)_1$ for ω_i does not change when referred to the frequency rather than amplitude modulation cases. Thus ω_i is readily calculated. By making reference to Figure 4 we try to choose the δ_i which will most closely give the same kind of gross characteristics as the actual power spectral density exhibits over the $(\varphi_i, \varphi_{i+1})$ interval. Now that δ_i has been selected we proceed in a similar manner to the method used in the derivation of relations (4.5). Thus we get

$$\omega_i = \frac{A}{P_i}$$

$$\Omega_i = \frac{\bar{X}\, J^o(\delta_i)}{J^2(\delta_i) Q_i} \qquad (4.7)$$

$$A_i = \frac{Q_i}{J^o(\delta_i)}$$

where we have used the notation of equations (4.6) and

$$J^0(\delta_i) = J_0^2(\delta_i) + 2\sum_{n=1}^{\infty} J_n^2(\delta_i)$$

$$J^2(\delta_i) = 2\sum_{n=1}^{\infty} n^2 J_n^2(\delta_i) \qquad (4.8)$$

By use of the relation

$$J_0(z) = \sum_{-\infty}^{\infty} J_n(z_1)J_n(z_2)\, e^{in\theta}$$

$$z^2 = z_1^2 + z_2^2 - 2z_1z_2 \cos\theta$$

we can show that

$$J^0(\delta) = 1$$

$$J^2(\delta) = \frac{\delta^2}{2}$$

Thus through equations (4.7) we can determine the parameters associated with a frequency modulated signal to approximate given power spectrum on an interval $(\varphi_i, \varphi_{i+1})$.

As can be seen by comparing Figure 4 with Figure 3, the use of frequency modulation is capable of giving much wider distributions of the power over the interval.

V. RANDOMLY SWITCHED SIGNAL

Consider the form of the power spectrum of a pure tone siren, i.e.

$$x_k(t) = \sqrt{A_k}\cos\omega_k t$$

If this signal is turned on and off in a random fashion we will have

$$x_k(t) = \frac{\sqrt{A_k}}{2}(1 + Q_k(t))\cos\omega_k t \qquad (5.1)$$

where $Q_k(t)$ is a random function assuming values +1 or -1, as illustrated in Figure 7.

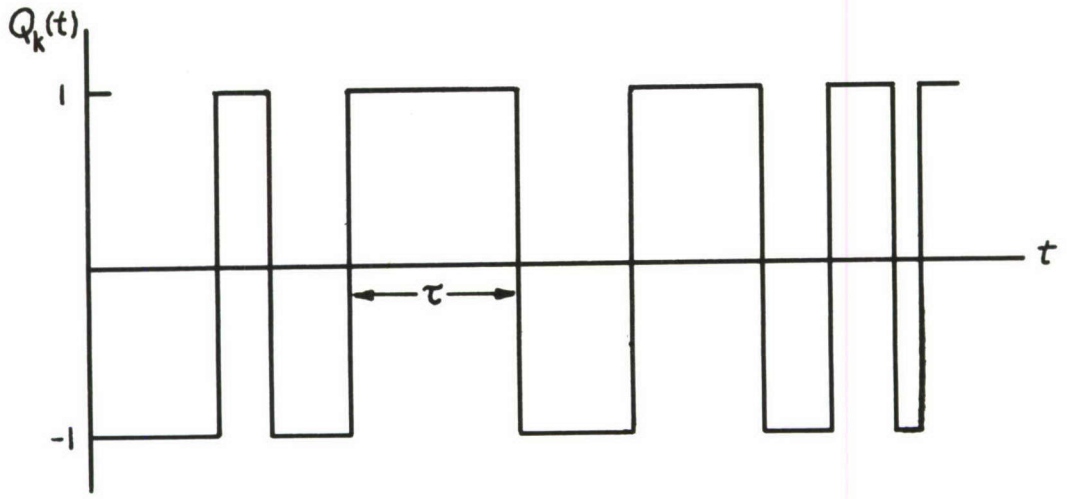


Fig. 7 - Random on/off amplitude modulation function

We denote the mean switching time, i.e., the expected time between changes in value, of $Q_k(t)$ by $\bar{T}_k$. If we assume that the period between switchings, i.e. τ in Figure 7, is a Poisson distrubuted random variable, then by a straightforward calculation, we find that the power spectral density of the signal (5.1) is given as

$$P_k(\omega) = \frac{A_k}{4}\{S_k(\omega-\omega_k) + S_k(\omega+\omega_k)\} \tag{5.2}$$

where

$$S_k(\omega) = \frac{\bar{T}_k}{\pi}\left\{\frac{1}{1 + \left(\frac{\omega\bar{T}_k}{2}\right)^2}\right\}$$

Note that this is the first time that we have been lead to a non-discrete power spectral density. This non-discrete form is realized due to the random nature of $Q_k(t)$. In previous methods we have always been concerned with deterministic forms (see section IV) which necessarily lead to discrete spectra. Since we are only concerned with power spectra that are symmetric about $\omega = 0$, we write $P(\omega)$ as

$$P(\omega) = \frac{1}{2} S_k(\omega) = \frac{\bar{T}_k}{2\pi}\left\{\frac{A_k}{1 + \left[\frac{(\omega-\omega_k)\bar{T}_k}{2}\right]^2}\right\} \tag{5.3}$$

This amounts to a folding about $\omega = 0$ of the spectrum. Figure 8 shows a typical form for $P(\omega)$ as given by (5.3)

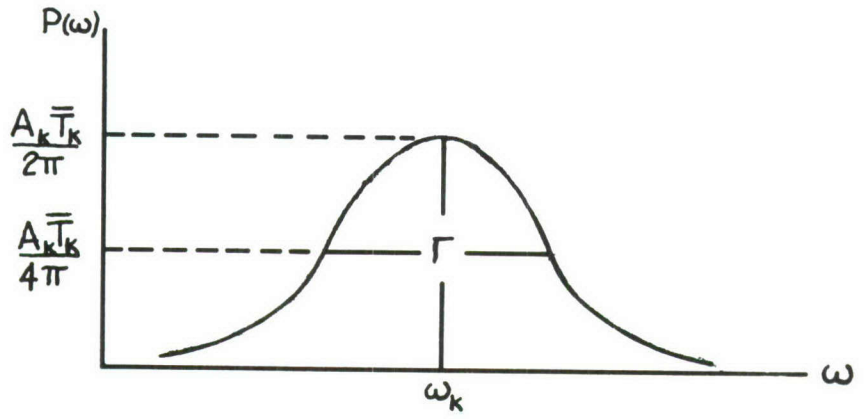


Fig. 8 - Power spectrum of a randomly switched pure tone signal

The half height width of the curve, as shown in Figure 8, is given as

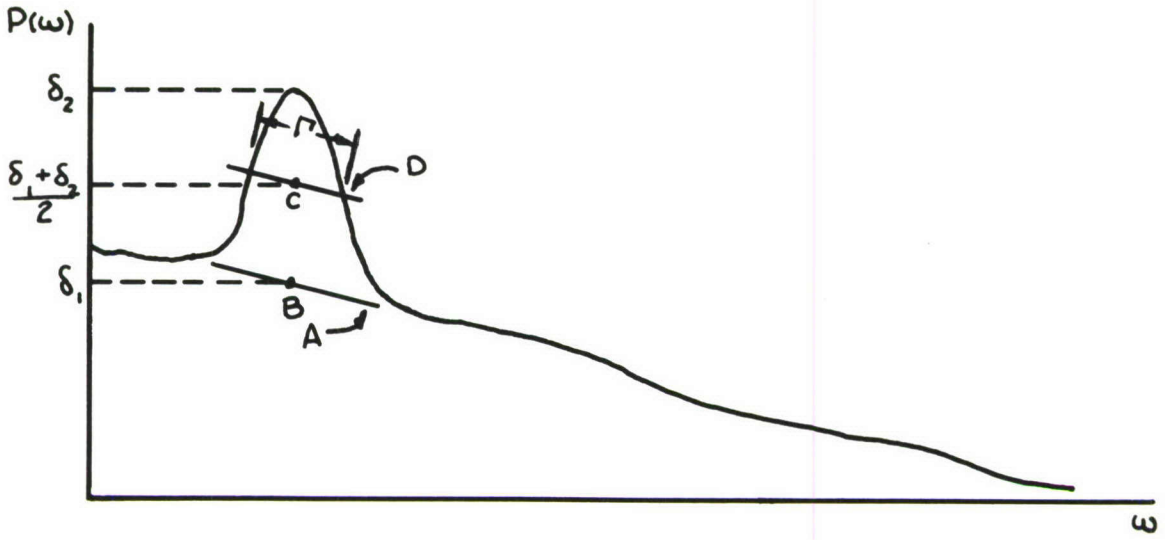
$$\Gamma = \frac{4}{\bar{T}_k}$$

This shows that the width of the spectrum increases and the height decreases as the mean switching time decreases.

One could now set up a least square method for the approximation of a given spectrum by a system of on-off modulated sirens. This, however, would be a rather involved scheme, since each siren involves 3 parameters (ω_k, $\bar{T}_k$, A_k) and we are lead to a system of nonlinear equations for their determination. When we examine a particular spectrum, there will quite probably exist certain resonances that are similar in appearance to the one illustrated in Fig. 9. Suppose Fig. 9a represents the spectrum we are interested in approximating. We wish to remove the resonance by approximating this portion of the power spectrum by an on-off modulated signal. We go through the following steps to find the values of the parameters:

(1) Draw a straight line, A, approximating the background under the peak.

(2) Determine the maximum, δ_2, value obtained and δ_1, the value of P at the point B. B is roughly half way under the peak on the line A.



(a) - A typical spectrum showing a resonance

(b) - Modified spectrum showing the removal of the resonance by means of an on/off modulated siren

Fig. 9

(3) Draw a line, D, through the point C, parallel to the line A. The point C is determined by $P = (\delta_1+\delta_2)/2$ and placed half way between the sides of the peak.

(4) Measure the width, Γ, of the peak on the line C, and redetermine the point C to be in the center of the peak.

(5) Determine the parameters of the siren as

$$\bar{T}_k = \frac{4}{\Gamma_k}$$

$$\omega_k = \omega \text{ of the point C} \qquad (5.4)$$

$$A_k = \frac{2\pi\,(\delta_2-\delta_1)}{\bar{T}_k}$$

By assigning the parameters as given in Eqn. (5.4) to one of the sirens we are left with a spectrum as illustrated in Figure 9b. This reduced spectrum is then more likely to be easily approximated by some other method.

The same procedure with a little more freedom of choice can also be used for two overlapping peaks where the maxima are distinct; but close together.

VI. GENERAL REPRESENTATION OF THE RANDOM SIGNAL BY APPROXIMATION OF THE POWER SPECTRUM BY RATIONAL POLYNOMIALS

In a report by Samuels [9] the problem of representing a random signal by a series of orthogonal functions with orthogonal random variables as coefficients has been studied. Here we shall be concerned with the solution presented there when the power spectral density, $S(\omega)$, has been approximated by a ratio of polynomials as follows:

$$S(\omega) = \frac{N((i\omega)^2)}{D((i\omega)^2)} = \frac{\sum_{h=0}^{q/2} b_h (i\omega)^{2h}}{\sum_{h=0}^{p/2} a_h (i\omega)^{2h}} \qquad (6.1)$$

The first computational problem associated with finding a solution is the approximation of the actual power spectral density by a form similar to equation (6.1). We have two major alternative methods by which we can attack this problem:

1) We can use a least square procedure in which we determine the constant a_k, b_k by minimizing the integral

$$Q(\underline{a},\underline{b}) = \int_0^\infty \left[S(\omega) - \frac{\sum b_k (i\omega)^{2K}}{\sum a_k (i\omega)^{2K}} \right]^2 W(\omega) d\omega$$

with respect to each of the a_k and b_k . $W(\omega)$ is the weighting function and $S(\omega)$ is the actual power spectral density.

2) We can use a minimax procedure in which the coefficients a_k and b_k are determined by minimizing the difference between the actual function and its approximation.

The least squares method is presented in detail in Appendix 1, while the Minimax procedure is presented in Appendix 2. While the least squares method gives an over all measure of the fit of the approximating function, one is not assured that very large errors will not be present in localized areas. The Minimax procedure remedies the objectionable character of the least squares method, in that we are assured that it will not miss local structure by large amounts. However, we loose the possibility of approximating the function in a "best" sense over the total region with the particular solution generated. Thus a decision must be made as to which characteristic must be more faithfully reproduced: a) local characteristics while possibly losing overall fitting; or b) gross characteristics while possibly losing local effects.

Once the form of equation (6.1) has been determined, i.e. a_k and b_k are known, we can determine the orthogonal functions for the approximation of the random signal. The computational scheme that is used to determine the parameter of this representation is given by the following six steps:

a) Solve the polynomial

$$\sum_{k=0}^{q/2} b_k \Omega_n^{2k} = |\lambda_n|^2 \sum_{k=0}^{p/2} a_k \Omega_n^{2k}$$

for the roots Ω_n . Note that there will be exactly p roots of which p/2 of these will be the negatives of the other p/2 roots. Denote these independent roots as Ω_{nk}, k = 1,2,...p/2 . It must also be recognized that we can not for the present determine Ω_{nk} except as a function of λ_n.

b) Determine the roots of

$$\sum_{k=0}^{p/2} a_k (i\omega)^{2k} = 0$$

These roots will come in pairs (one the negative of the other). Let us denote the roots for which the imaginary part is positive by ω_k, k = 1,... p/2 . The values of ω_k will not be dependent on λ_n, thus they are readily computed.

c) Find the quantities

$$B_k = \frac{N((i\omega_k)^2)}{\frac{d}{d\omega} D((i\omega)^2)\Big|_{\omega=\omega_k}} \quad , \quad k = 1,2,\ldots p/2$$

d) Solve the nonlinear system

$$\begin{vmatrix} \frac{1}{\Omega_{n1}-i\omega_1} & \frac{1}{\Omega_{n1}-i\omega_1} & \cdots & \frac{-1}{\Omega_{n\ell}+i\omega_1} & \frac{-1}{\Omega_{n\ell}+i\omega_1} & \cdots \\ \frac{1}{\Omega_{n1}-i\omega_2} & \cdots & & & & \\ \vdots & & & & & \\ \frac{1}{\Omega_{n1}-i\omega_{p/2}} & \cdots & & & & \\ \frac{e^{(\Omega_{n1}+i\omega_1)T}}{\Omega_{n1}+i\omega_1} & \frac{e^{(\Omega_{n1}+i\omega_o)T}}{\Omega_{n1}+i\omega_1} & \cdots & \frac{e^{(-\Omega_{n1}+i\omega_1)T}}{-\Omega_{n\ell}+i\omega_1} & & \cdots \end{vmatrix} = 0 \qquad (6.3)$$

for λ_n, where we have substituted the expressions obtained for Ω_{nk} in section a). This is the equation 30 of [9]. Number the solutions according to the index n. Since (6.3) will be a nonlinear equation, there may possibly be an infinity of solutions.

e) Solve the system of linear equations.

$$\sum_{k=1}^{p/2} B_S \left\{ \frac{A^+_{nk}}{\Omega_{nk}-i\omega_k} + \frac{A^-_{nk}}{-\Omega_{nk}-i\omega_k} \right\} = 0 \quad S = 1,\ldots p/2$$

$$\sum_{k=1}^{p/2} B_S \left\{ \frac{A^+_{nk} e^{(\Omega_{nk}+i\omega_S)T}}{\Omega_{nk}+i\omega_n} + \frac{A^-_{nk} e^{(-\Omega_{nk}+i\omega_S)T}}{-\Omega_{nk}+i\omega_n} \right\} = 0$$

$$S = 1,\ldots,p/2$$

for each A^+_{nk} and A^-_{nk} given each of the λ_n determined in section d). Note that one of the unknowns is not determined. Let this one be $A_{np/2}$. Then each of the $A^+_{nk}/A^+_{np/2}$, $k = 1,\ldots p/2 - 1$ and $A^-_{nk}/A^+_{np/2}$, $k=1,\ldots p/2$ is known for each λ_n.

f) Given all the foregoing we then have

$$\psi_n(t) = \sum_{k=1}^{p/2} \{A_{nk}^{+} e^{\Omega_{nk}t} + A_{nk}^{-} e^{-\Omega_{nk}t}\} \qquad (6.4)$$

By forcing ψ_n to be unitary, i.e..

$$\int_0^T \psi_n(t)\, \psi_n^*(t)dt = 1$$

we determine $A_{np/2}^{+}$.

Thus (6.4) represents the system of orthonormal functions necessary to approximate the given random signal $x(t)$ as

$$x(t) = \sum_{n=0}^{\infty} \theta_n \psi_n(t)$$

where θ_n are orthogonal random variables.

It takes little examination to see that this will require a very lengthy computation for reasonably sized p and q.

To illustrate the application of the method of rational polynomial approximation in obtaining the orthogonal function, we consider the simplest case possible, namely: $p = 1$ and $q = 0$. Thus $S(\omega)$ is approximated by

$$S(\omega) = \frac{2A\beta}{\beta^2+\omega^2}$$

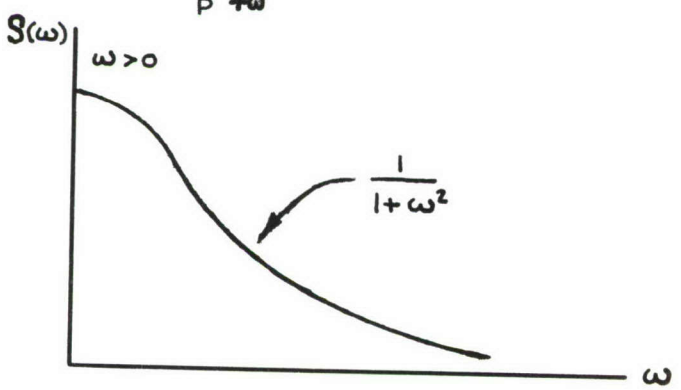


Fig. 10 - Power spectrum of a signal with a Markoff auto-correlation function

This is the power spectral density of a process with a Markoff autocorrelation function illustrated in Fig. 10. After some manipulation (see [9] p. 17-20) we find that

$$\omega_n \cos\left(\frac{\omega_n T}{2}\right) + \beta \sin\left(\frac{\omega_n T}{2}\right) = 0$$

$$\omega_n = +\sqrt{\frac{2\beta A}{|\lambda_n|^2} - \beta^2} \tag{6.5}$$

Thus the λ_n are determined and we can write ψ_n as

$$\psi_n(t) = \frac{2ia_n}{\beta+i\omega_n}\left\{\omega_n \cos \omega_n t + \beta \sin \omega_n t\right\}$$

where a_n is determined as

$$|a_n|^2 = \left|\frac{\beta+i\omega}{2}\right|^2 \left\{\int_0^T \left|\omega_n \cos \omega_n t + \beta \sin \omega_n t\right|^2 dt\right\}^{-1}$$

VII. APPLICATION TO A PARTICULAR SIGNAL

As an illustration of the methods presented in the previous sections, we now give several applications to a particular signal.

Figure 11 illustrates the representative trace of the random signal that has been digitalized and used in this analysis. From this signal we have calculated its amplitude probability function, see Fig. 12, its covariance, see Fig. 13 and its power spectral density, see Fig. 14, according to the methods of section II. It is to be noted that the power spectral density is quite ill-behaved. This is due to the nature of the random noise generator used. In practice one would expect a less harsh curve. For some of the analysis we have used a "Smoothed" power spectrum which would probably be closer to the types of spectra physically measured. This is exhibited in Figure 15. Note that there exist marked resonances, however, they are not as pronounced. Also the rapid low level background has been filtered out. When this power spectrum is used it will be clearly noted.

A. Frequency and Amplitude Modulation

As in our first illustration, let us suppose that we wish to approximate the power spectral density with the pure tone sirens only. Then according to the equal energy partitioning and the moment matching technique of section IV we can easily compute the operating frequencies and powers of each siren. Again we wish to point out that the equal energy method of assigning these parameters to each unit leads to realizable power assignments to each unit and in addition offers an intuitively rational method of selection. It is rational in that one of the foremost considerations of approximating a random signal is that the energies associated with particular frequency regions be as faithfully reproduced as possible. This assures that the response characteristics of any system which uses the approximate signal as an input is as close to its response when subjected to the physically occurring signal.

One of the questions raised is: how many pure tone sirens are necessary to simulate the signal. As an illustration, a number of cases have been carried out on the power spectrum of Figure 14. The following list gives the number of pure tone sources, the table on which the parameters are exhibited, and the appropriate figure on which a comparison is made between the approximation, the actual or smoothed power spectra.

Case No.	No. of sirens		Table*	Figure
	High	Low		
1	25	9	1	16
2	20	6	2	17
3	15	6	3	18
4	10	6	4	19
5	5	4	5	20

*See Appendix D.

Along the abscissa of the Figures 16 through 20 the vertical lines represent the center frequencies at which the sirens function. The straight line segments represent the power of the siren smoothed over its interval of operation, and the irregular line indicates the real power spectrum. It is clear from Figures 16 and 17, that cases 1 and 2 approximate the actual spectrum quite closely. Cases 3 and 4 although not

approximating the actual power spectrum, do represent the smooth power spectrum rather well. The last case fails to give anything other than an indication of the gross power distribution and is completely devoid of any local structure. As another measure of the meaningfullness of these approximations, the amplitude probability density has been computed for case 1, see Figure 22. When compared to that of the original signal, we find that the fit is quite good except for the very high amplitudes. The tables associated with case 1 through 5 also give the appropriate frequency and amplitude modulation parameters for the intervals. We can use these parameters to distribute the energy over the interval in a manner that will more closely yield the approximate structure given by the figures. It should be clear through these examples that the number of sirens necessary is directly a function of the power spectrum and that no general rule can be stated.

B. Randomly Switched Pure Tone Signal

For the present consider the smoothed power spectrum of Figure 15. Note that there are five marked resonances indicated by the arrows. Each of these has structure similar to that of Figure 9a. We wish to remove these resonances by use of randomly switched pure tone sirens.

After following the procedure outlined in section V, we arrive at the following parameters for the five sirens required to remove these resonances:

Resonance No. k	ω_k	A_k	$\bar{T}_k$
1	3	289.0	1.000
2	18	314.1	.500
3	26.5	351.8	.571
4	48	326.7	.500
5	59.5	395.8	.222

When the parts of the spectrum are removed which are contributed by these sirens, we have left the spectrum represented in Figure 22 by the lower line, termed the resolved spectrum. The resolved spectrum, Figure 23, is much smoother and will be far easier to approximate with other methods than the spectrum we started with.

C. Rational Polynomial Fitting and Use of the Markoff Correlation.

By removing the resonances in the smoothest power spectrum by the methods of the previous subsection, we have arrived at the resolved spectrum, Figure 23. This has the same general shape as that of Figure 10 and thus suggests that it be approximated by the use of an orthogonal expansion in the sense of section VI. We find that a curve given by

$$P(\omega) = \frac{48600}{60^2 + \omega^2} \tag{7.1}$$

falls just below the resolved spectrum (see Figure 23). We choose such a curve so that the portion left after subtraction will be everywhere positive. The left-over can then be approximated by other means. Using the length of record analyzed, 10 sec, we find that eigenfrequency equation (6.5) is in the form

$$\frac{2\pi}{60}\omega_n + \tan(10\pi\omega_n) = 0 \tag{7.2}$$

where ω_n is now in cycles per second. It is clear that the roots ω_n of this equation are bounded as

$$\frac{n\pi}{2} < 10\pi\,\omega_n < \frac{\pi}{2}(n+2)$$

or

$$\frac{n}{20} < \omega_n < \frac{n}{20} + \frac{1}{10}$$

Thus the frequencies will be spaced at 20 per cycle. This points out the basic problem with the approximation of the actual signal by the orthogonal function scheme of section VI. Namely, that it leads to unwieldy numbers of sirens operating within the frequency regions of interest.

This same power spectrum, namely that of equation (7.1) can be approximated by the use of only one siren which is switched on and off in the manner of section V, if the parameters are chosen as

$$\omega \sim 0$$

$$\frac{}{T} = \frac{1}{30}$$

$$A = 8100\pi$$

Remembering that the signal of the randomly switched siren is of the form

$$x(t) = \frac{\sqrt{A}}{2}\,[1+Q(t)]\,\cos \omega t,$$

then the approximating siren has the modulation

$$x(t) = \frac{\sqrt{A}}{2}\,[1+Q(t)]$$

For physical operation, ω may have to be chosen to be small, but not zero. If this amplitude is outside the physical operating limits of one siren, then several sirens may be used to generate the amplitude, while all are similarly modulated with $(1 + Q(t))$. This type of operation also has its drawbacks in that it is doubtful that the very small mean switching time can be physically realized.

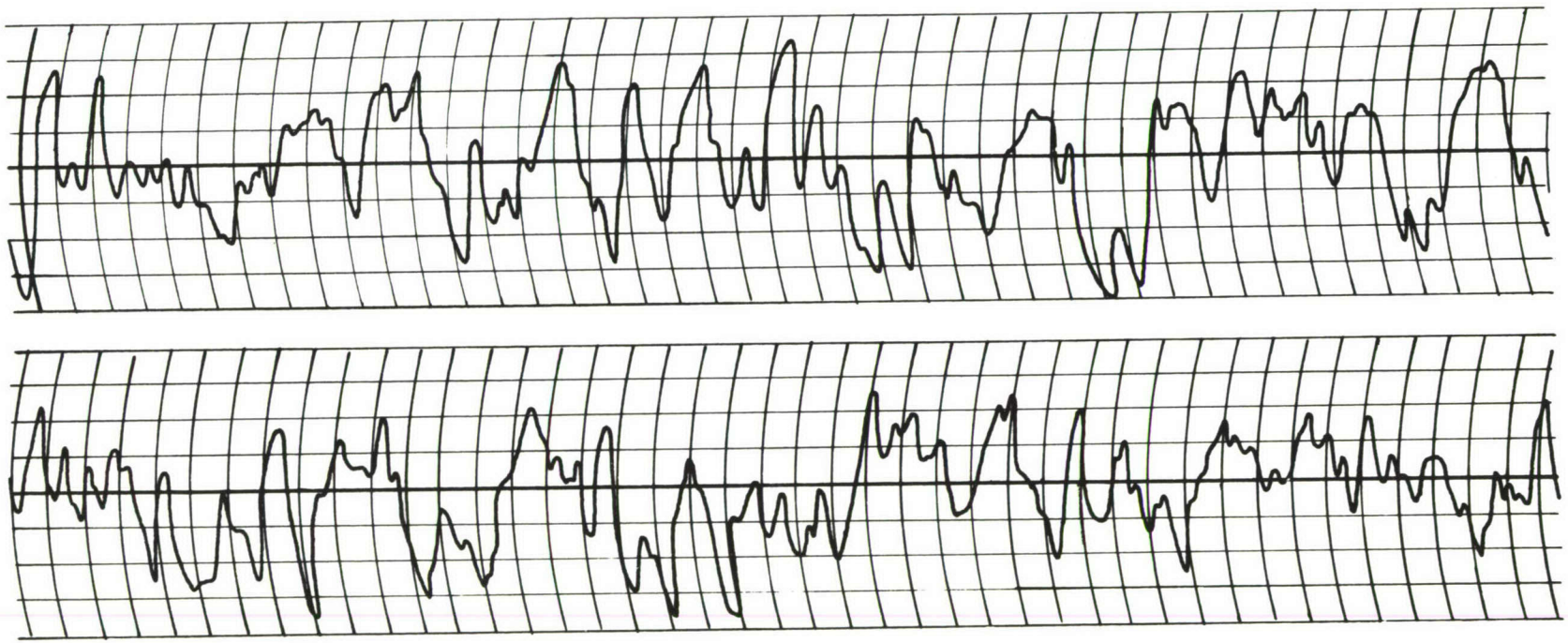


Fig. 11 Typical time track of the function used in Section VII

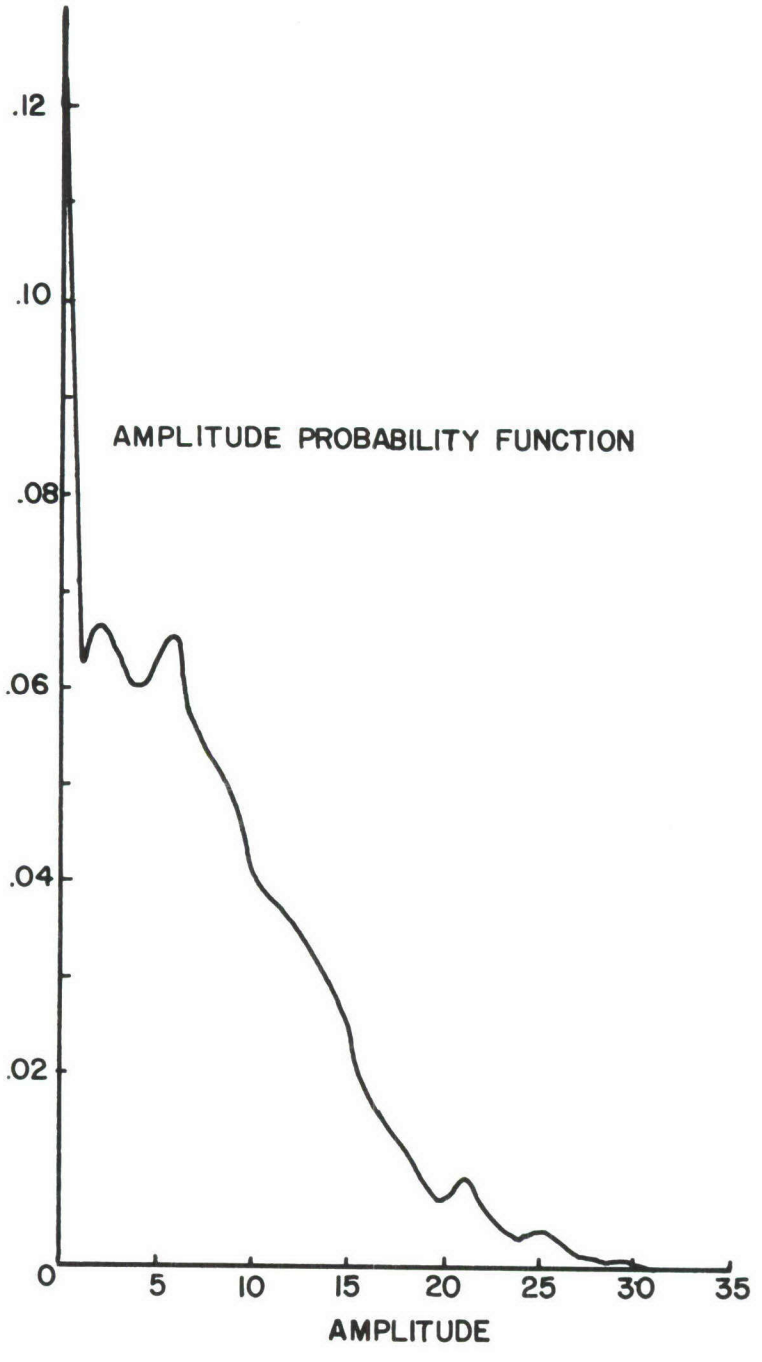


Fig. 12 Amplitude probability density for the sampled signal

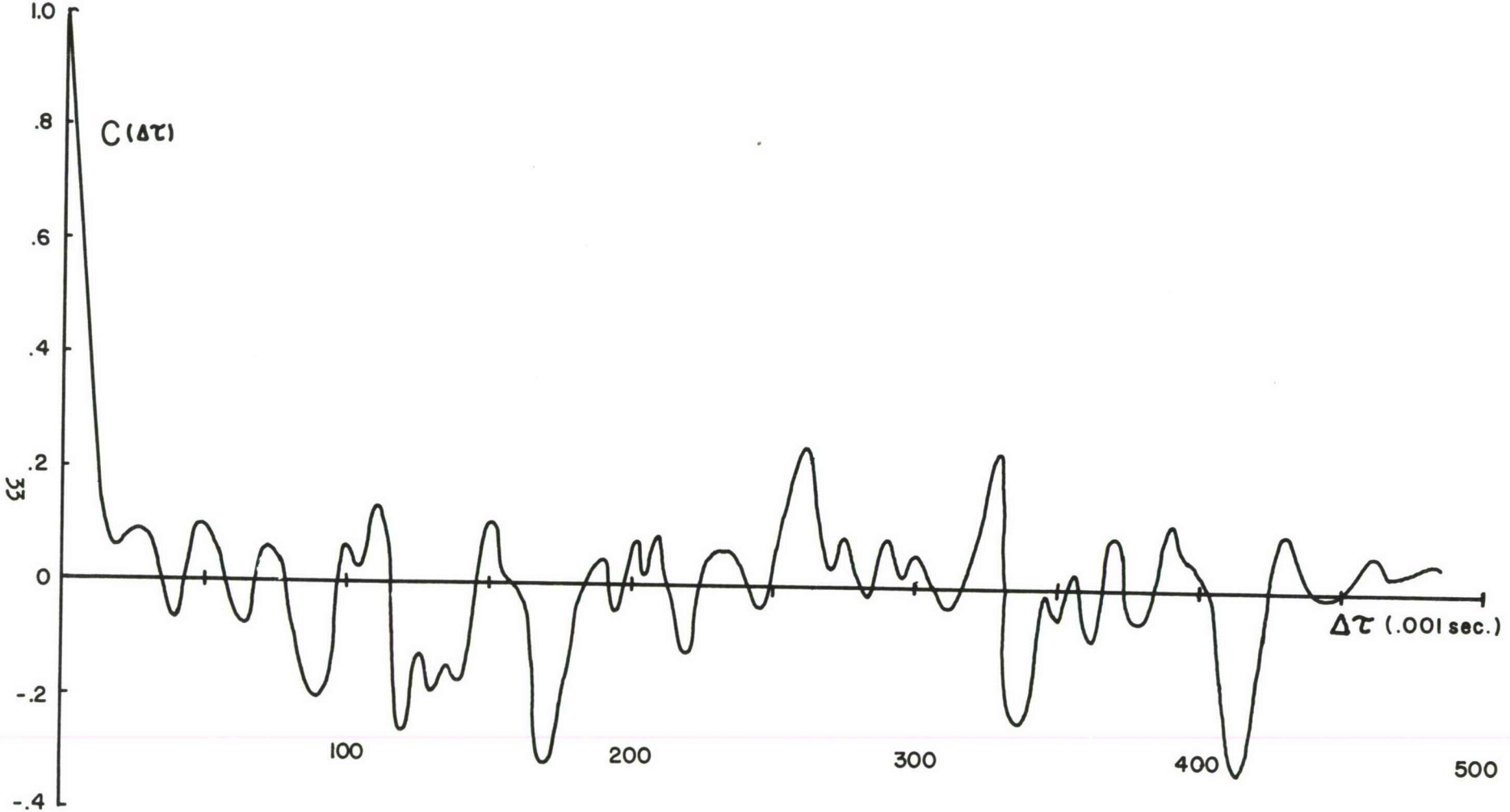


Fig. 13 Autocorrelation function for the sampled signal

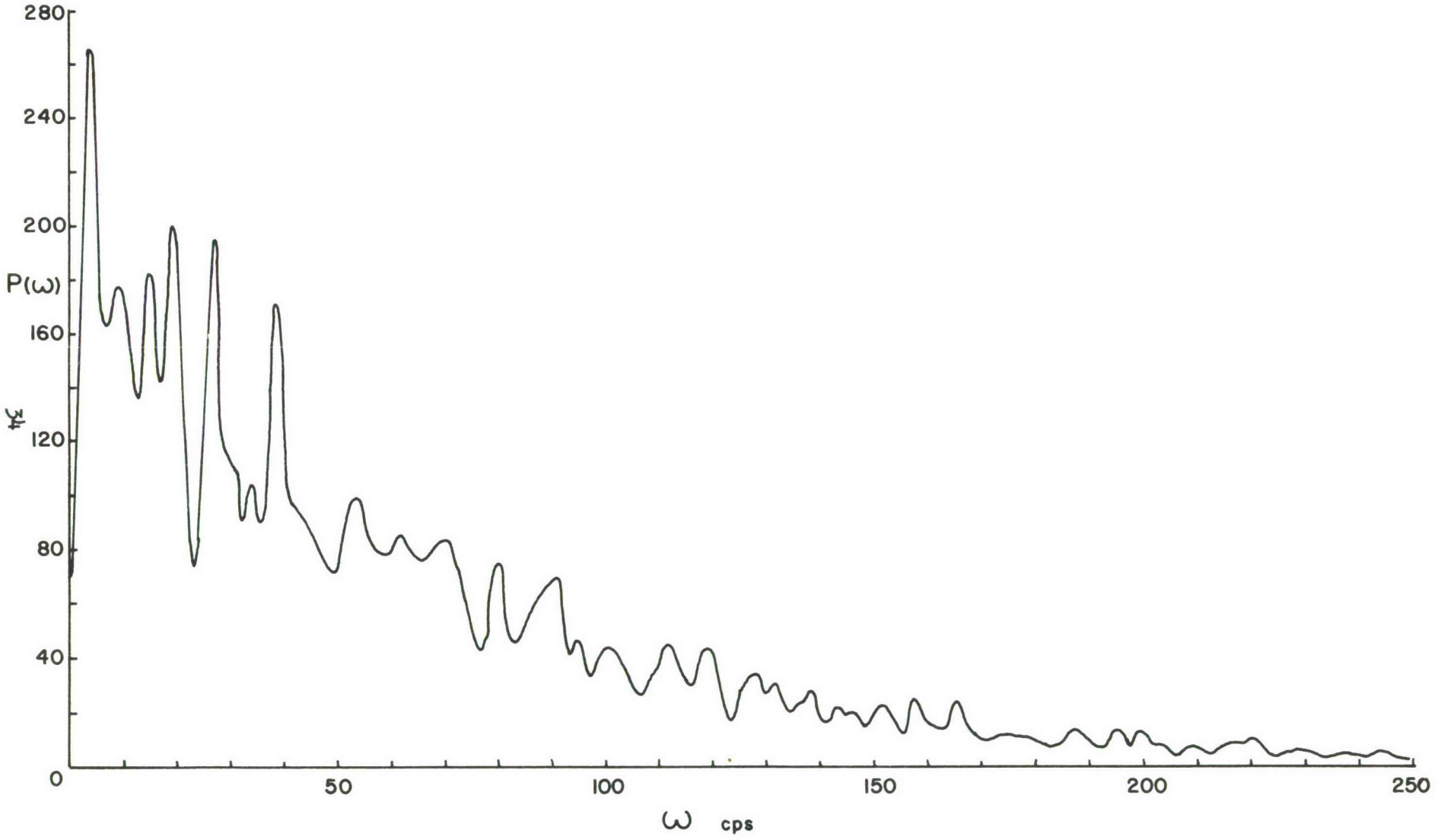


Fig. 14 Power spectral density of the sampled signal

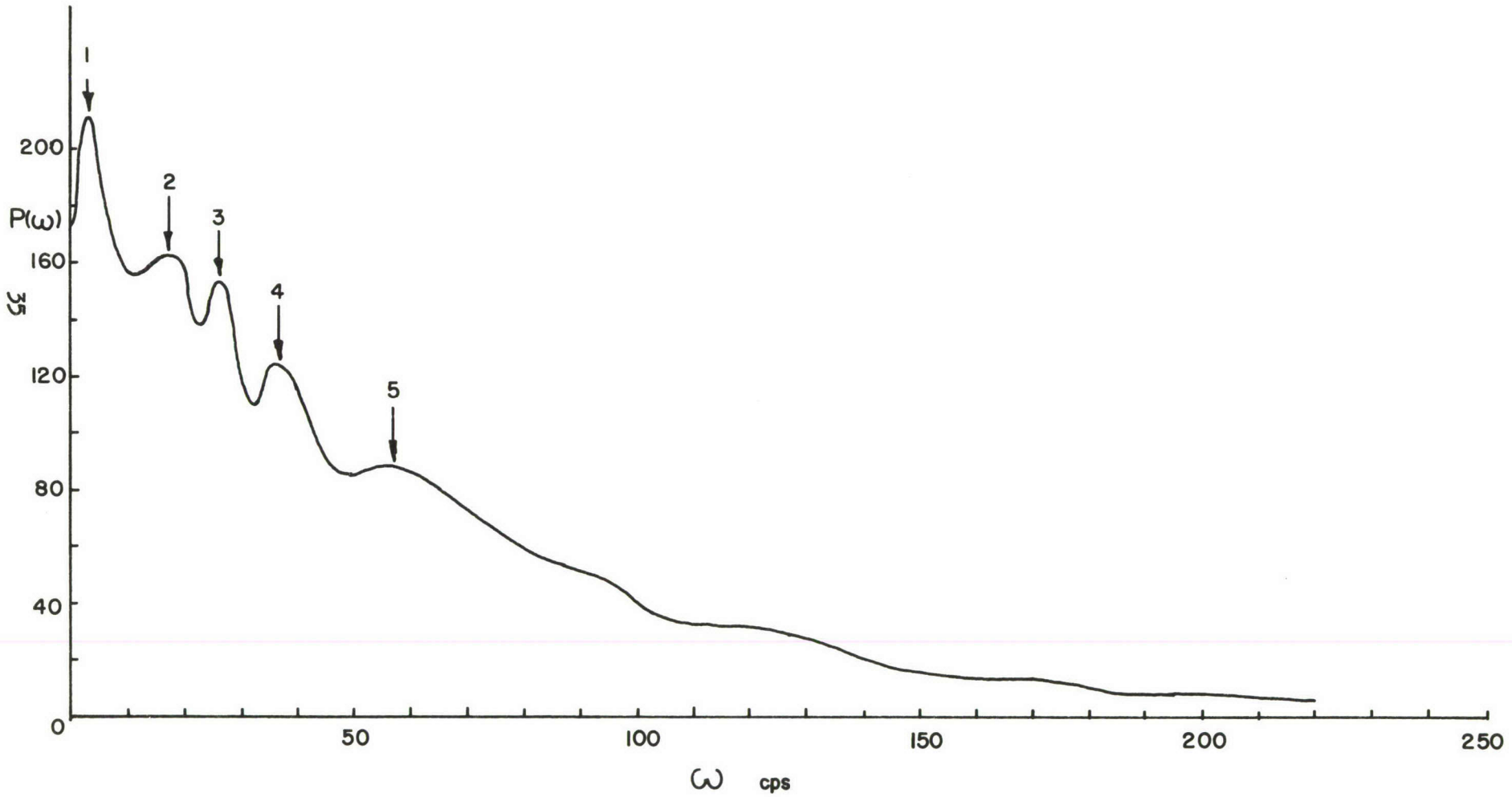


Fig. 15 Smoothed power spectral density

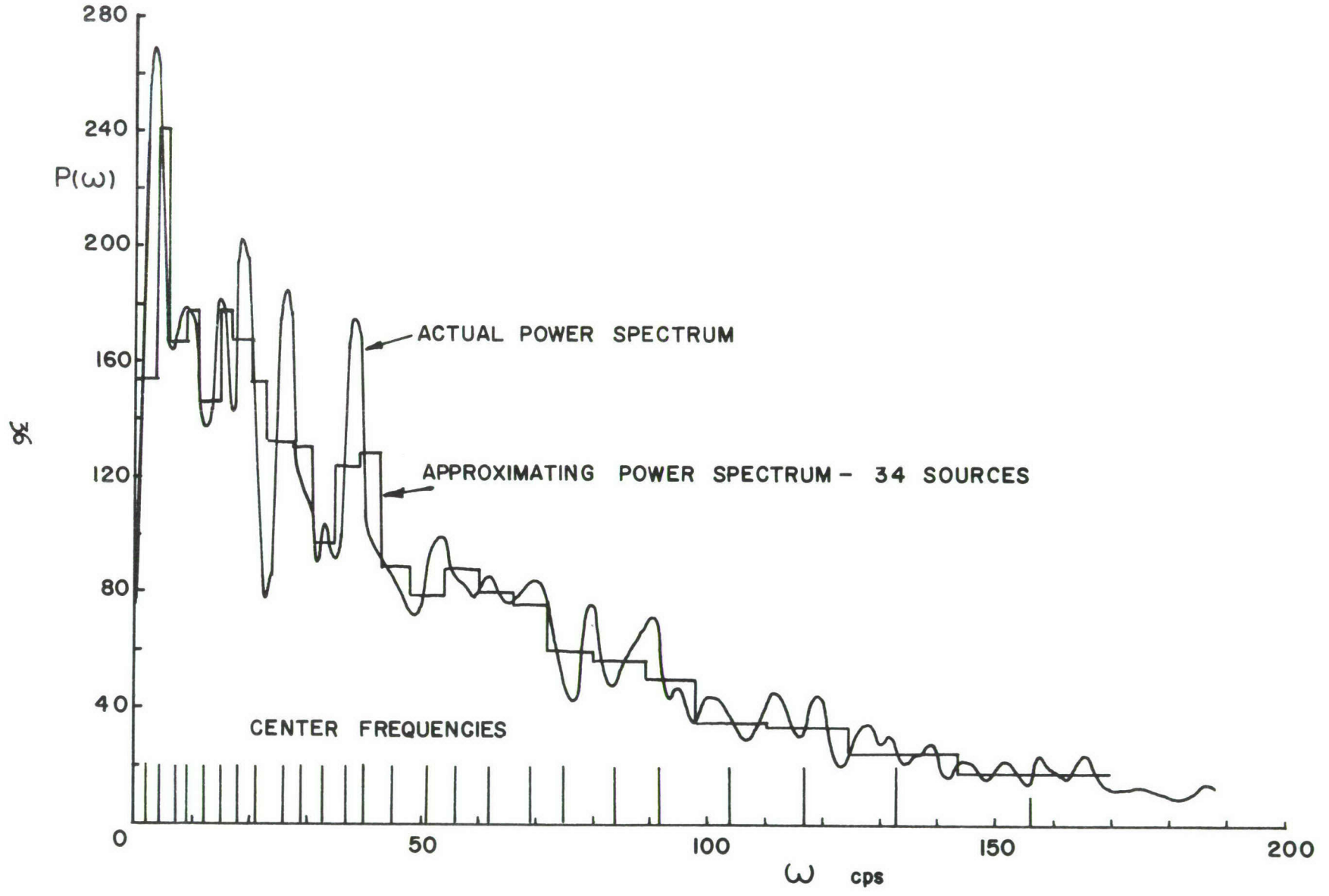


Fig. 16 Power spectral density approximation with 34 pure-tone sources

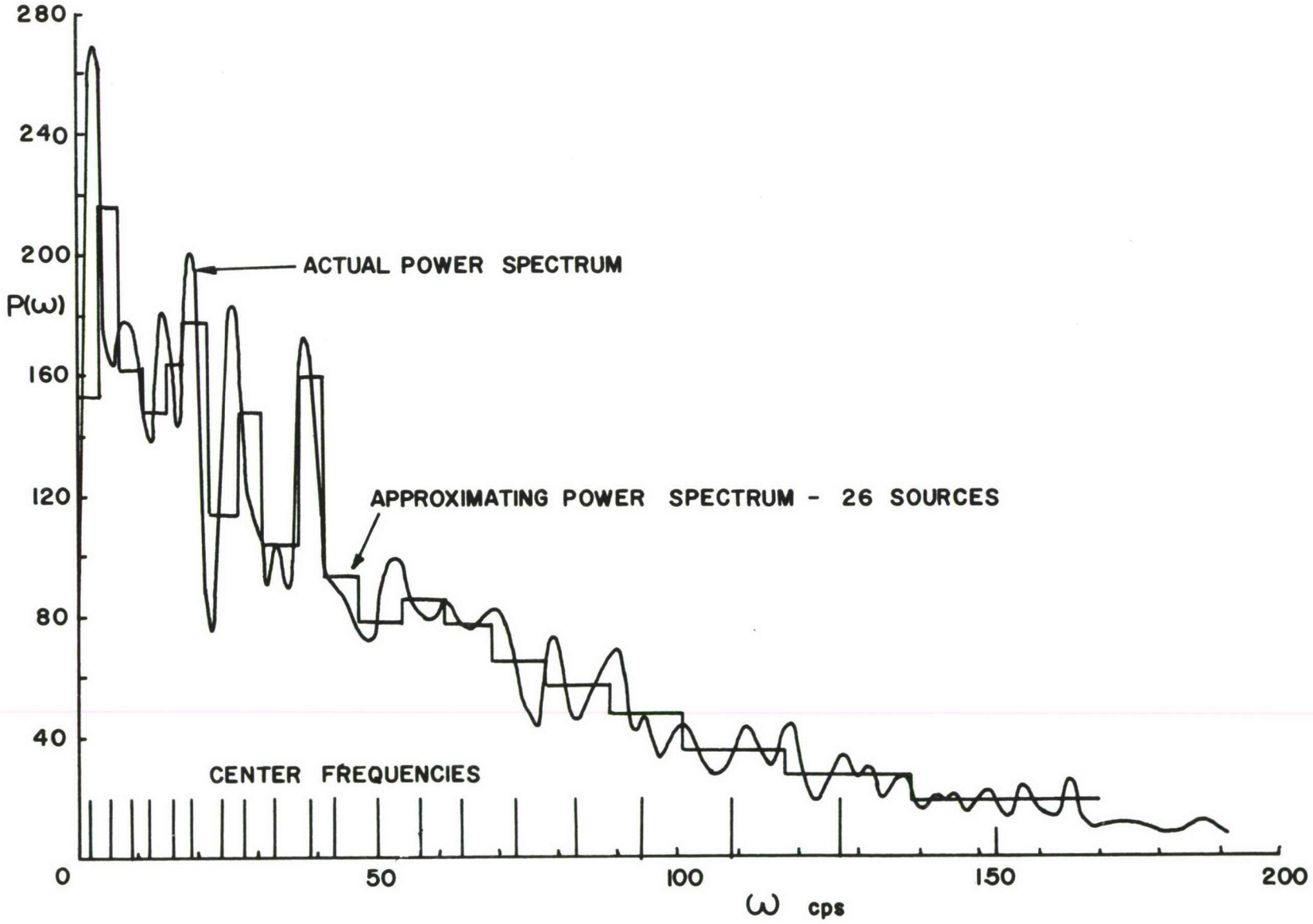


Fig. 17 Power spectral density approximation with 26 pure-tone sources

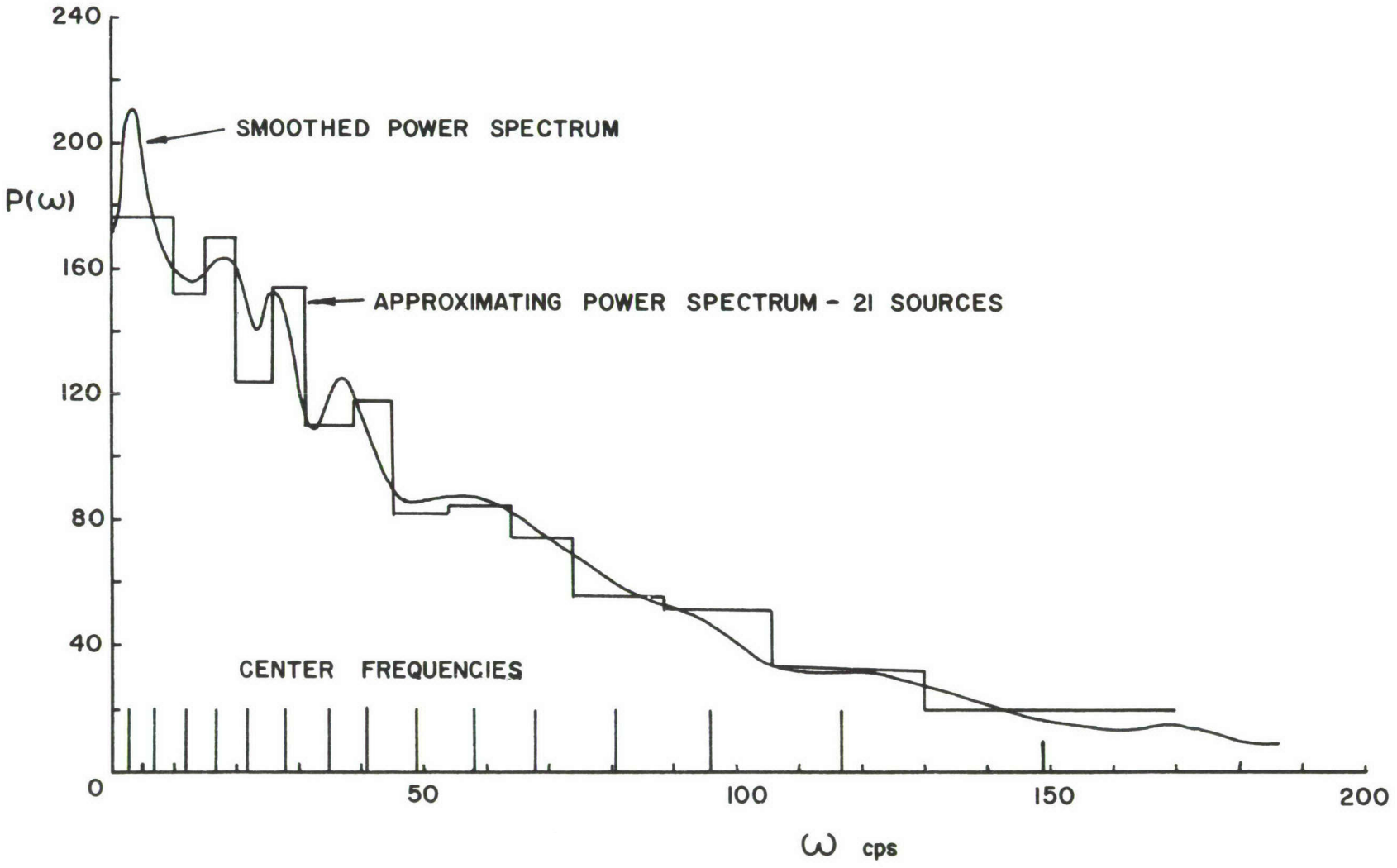


Fig. 18 Power spectral density approximation with 21 sources

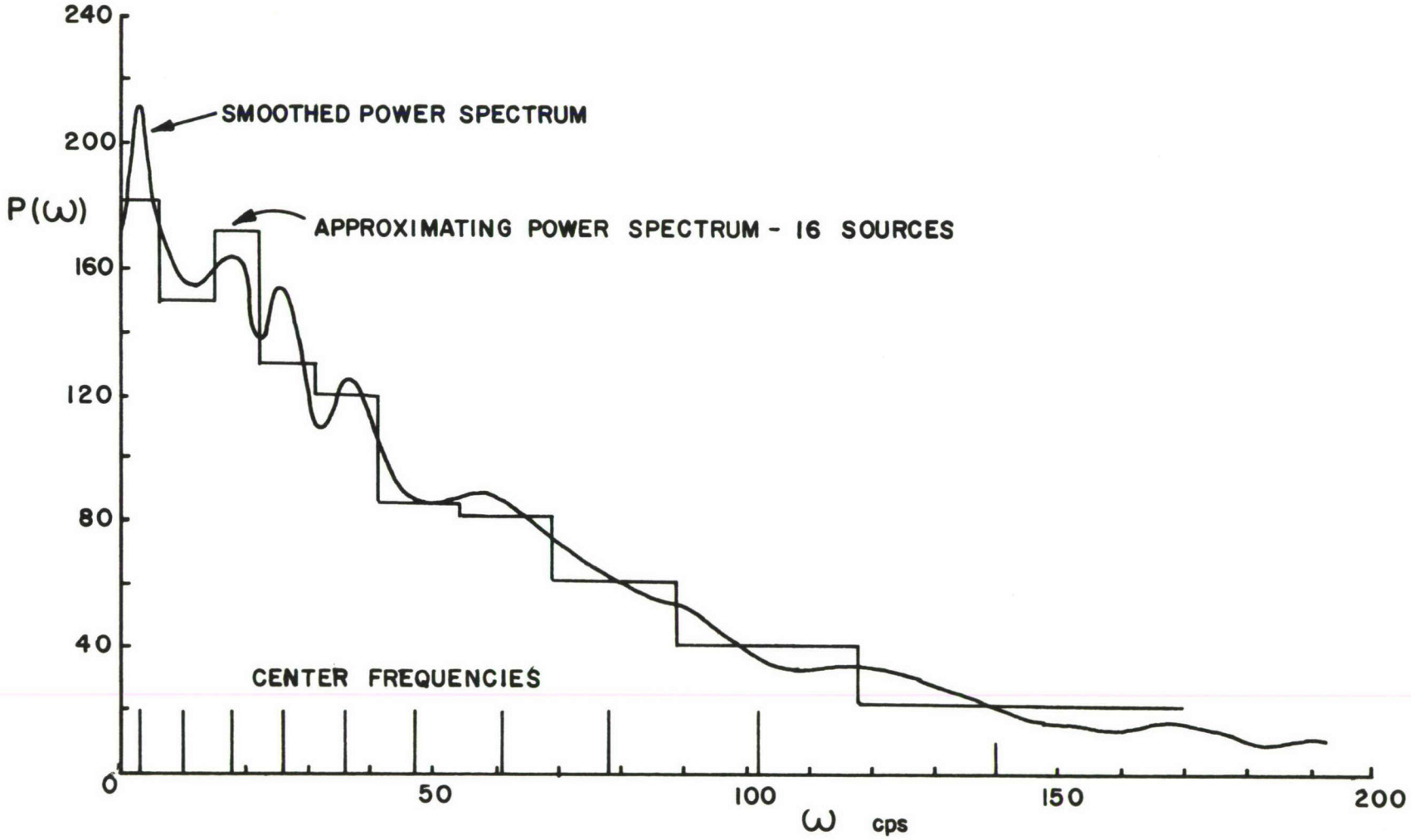


Fig. 19 Power spectral density approximation with 16 pure-tone sources

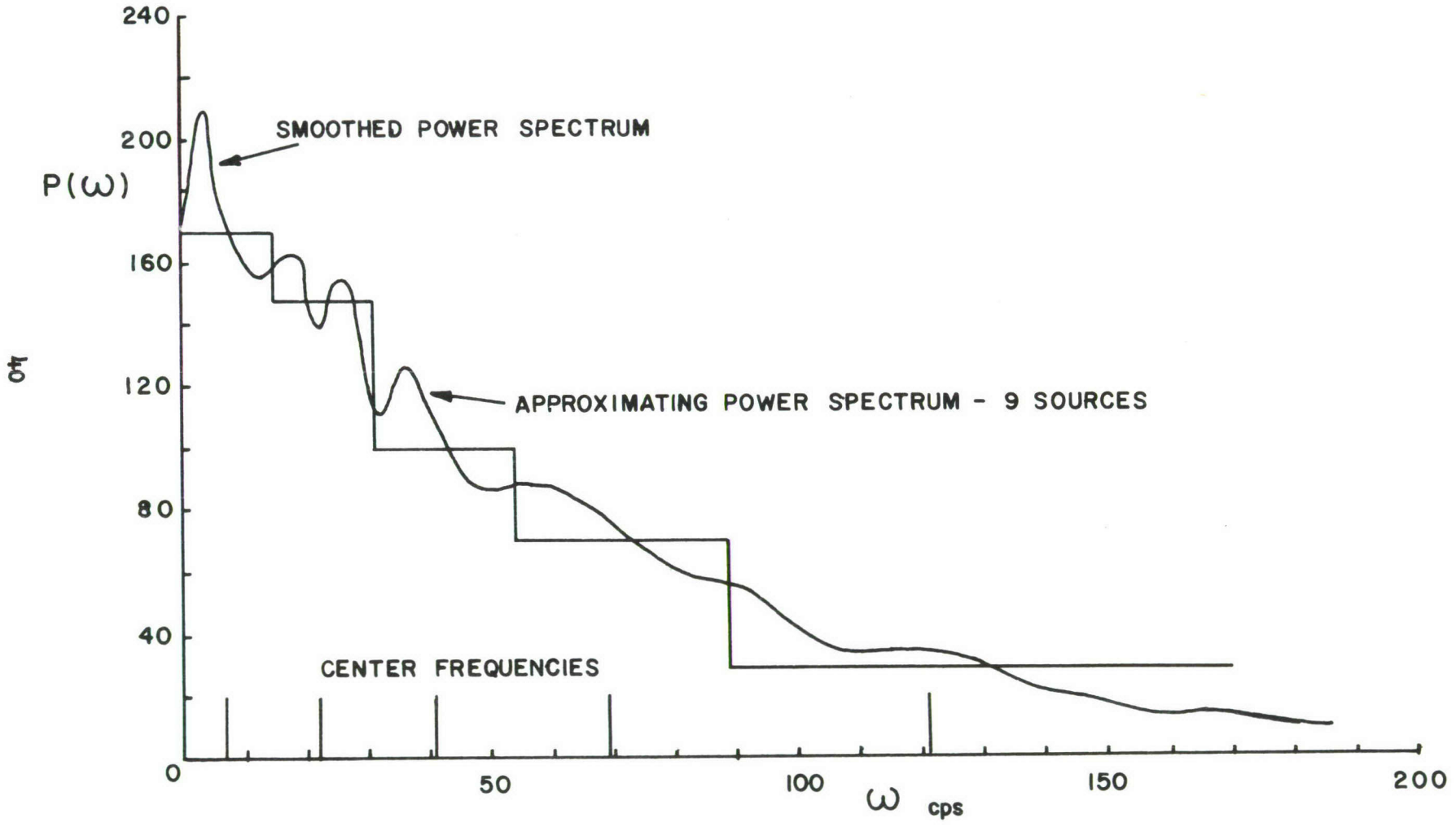
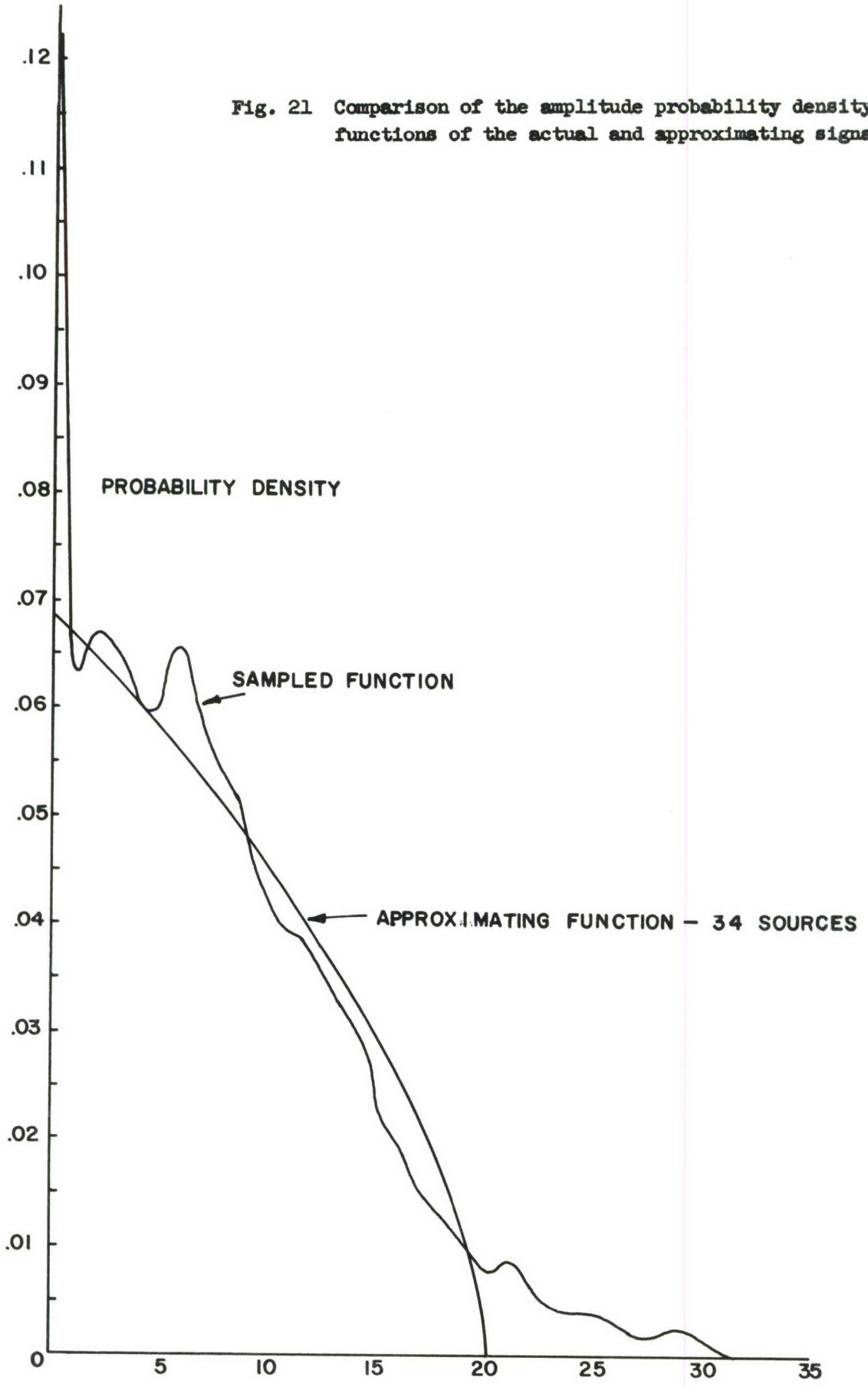


Fig. 20 Power spectral density approximation with 9 pure-tone sources

Fig. 21 Comparison of the amplitude probability density functions of the actual and approximating signals



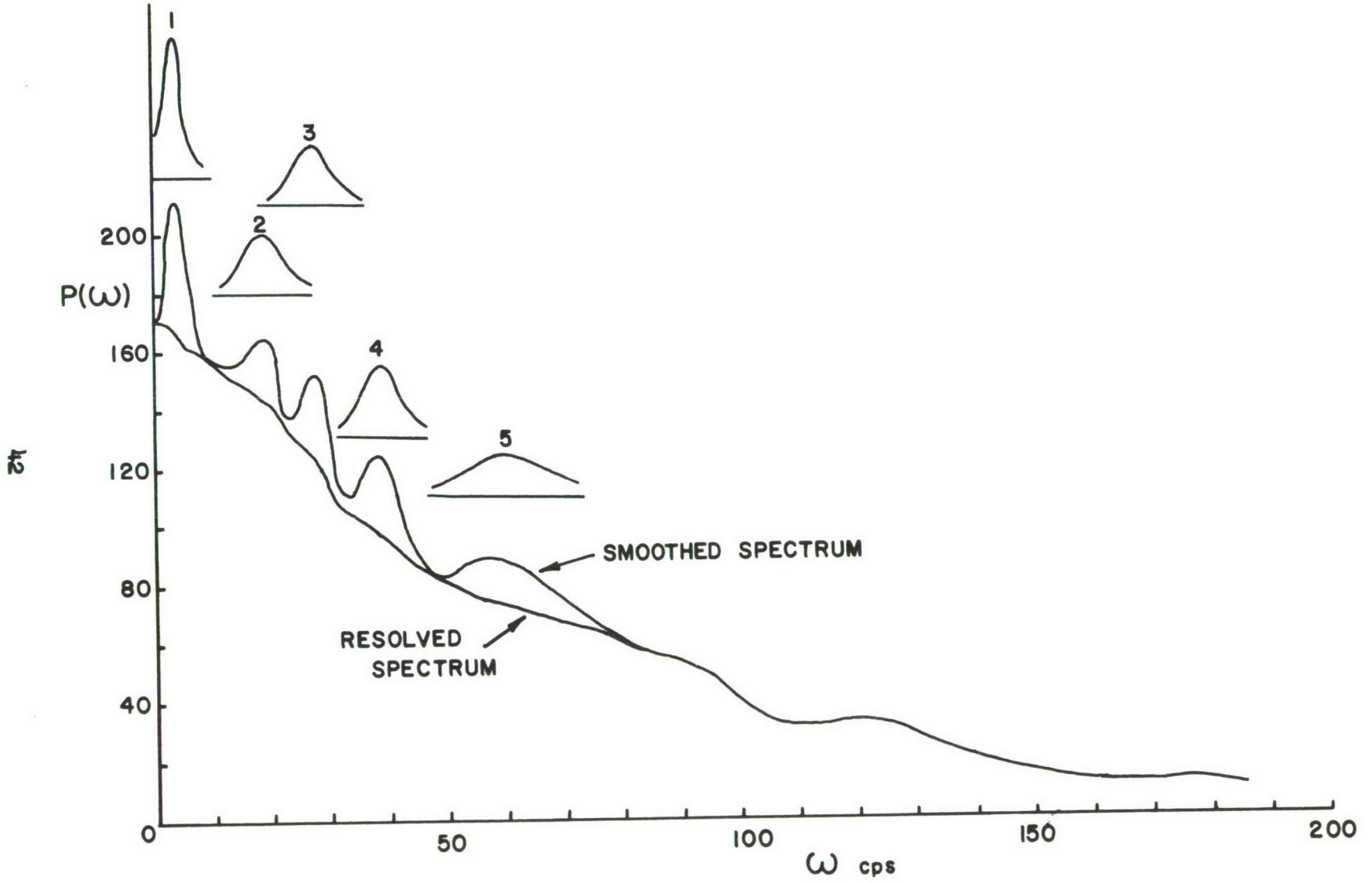


Fig. 22 Removal of resonances from the smoothed power spectral density

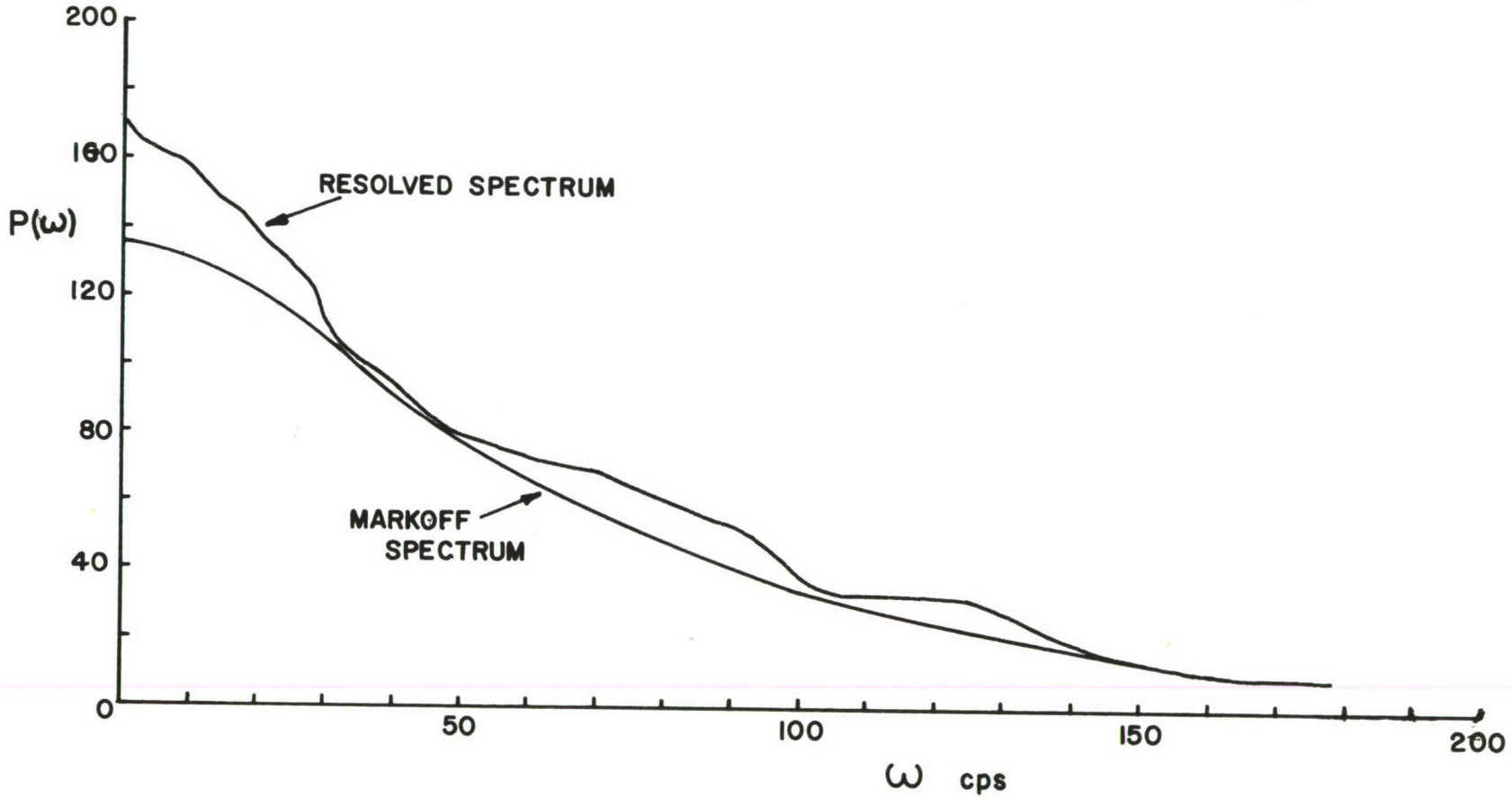


Fig. 23 Approximation of the resolved spectrum by a process with a Markoff autocorrelation function

Appendix A

Least Squares Determination of Rational Polynomials Coefficients

The problem which we wish to consider is that of approximating a given tabulated function f(x) in terms of a ratio of two polynomials in x, i.e., we construct a function, $f_N(x)$, of the form

$$f_N(x) = \frac{\sum_{i=0}^{p} \alpha_i x^i}{\sum_{i=0}^{q} \beta_i x^i}$$

such that $f_N(x)$ is the "best" (in some specified sense) approximation of f(x).

In this appendix we wish to examine the problem of the determination of the coefficients α_i, β_i, under the so-called "least squares" criterion. As usually formulated, the least squares method calls for the minimization of the function

$$Q(\underset{\sim}{\alpha},\underset{\sim}{\beta}) = \int_0^\infty [f(t) - f_N(t)]^2 \, W(t)dt$$

with respect to the parameters $\underset{\sim}{\alpha}$, $\underset{\sim}{\beta}$, where W(t) is some non-negative weighting function which many authors set arbitrarily to unity. Forming $Q(\underset{\sim}{\alpha},\underset{\sim}{\beta})$ using our supposed $f_N(t)$ we have

$$Q(\underset{\sim}{\alpha},\underset{\sim}{\beta}) = \int_0^\infty \left[f(x) - \frac{\sum \alpha_i x^i}{\sum \beta_i x^i}\right]^2 W(x)dx \qquad \text{(A.1)}$$

For $Q(\underset{\sim}{\alpha},\underset{\sim}{\beta})$ to be minimized with respect to the parameters α_i, β_i, it is necessary that

$$\frac{\partial Q}{\partial \alpha_i} = 0 \qquad i = 0,1,\ldots p$$

$$\frac{\partial Q}{\partial \beta_i} = 0 \qquad i = 0,1,\ldots q \qquad \text{(A.2)}$$

Of course the solution for (A.2) must be further examined to ascertain which solution actually gives the minimum. Performing the indicated differentiation on $Q(\alpha,\beta)$ as given in (A.1) it is seen that it is desirable to choose a weighting function $W(x)$ such that we can remove the unknowns β_i from the denominator. By this consideration we are led to make the choice of $W(x)$ as

$$W(x) = \left[\sum_{i=0}^{q} \beta_i x^i\right]^2 e^{-Kx} \tag{A.3}$$

where K is an arbitrary positive predetermined constant. We propose to call this choice the denominator weighted least squares. The reason for including the term e^{-Kx} will be apparent later. Clearly, $W(x)$ as given in (A.3) is non-negative. $Q(\alpha,\beta)$ can then be written as:

$$Q(\alpha,\beta) = \int_0^\infty \left[f(x)\sum \beta_i x^i - \sum \alpha_i x^i\right]^2 e^{-Kx} dx \tag{A.4}$$

We first observe that if $f(x)$ is bounded as $x \to \infty$, then we must require that $p < q$ for (A.4) to be bounded. Also we note that the system of parameters (α,β) can be determined only within an arbitrary multiple. Thus without loss in generality we set

$$\alpha_o = 1$$

We have

$$\frac{\partial Q}{\partial \beta_j} = 2\int_0^\infty \left\{f(x)\sum_0^q x^i\beta_i - \sum_0^p \alpha_i x^i\right\} f(x)x^j e^{-Kx} dx$$

or

$$\sum_{i=0}^{q} \beta_i \int_0^\infty f^2(x)x^{i+j} e^{-Kx} dx - \sum_{i=1}^{p} \alpha_i \int_0^\infty f(x)x^{i+j} e^{-Kx} dx \tag{A.5}$$

$$= \int_0^\infty f(x)x^j e^{-Kx} dx$$

for $j = 0,1,\ldots q$

and

$$\frac{\partial Q}{\partial \alpha_j} = 2\int_0^\infty \{f(x) \sum_0^q \beta_i x^i - \sum_0^p x^i \alpha_i\} x^j e^{-Kx} dx$$

or

$$\sum_{i=0}^{q} \beta_i \int_0^\infty f(x) x^{i+j} e^{-Kx} dx - \sum_{i=1}^{p} \alpha_i \int_0^\infty x^{i+j} e^{-Kx} dx = \int_0^\infty x^j e^{-Kx} dx \qquad \text{(A.6)}$$

for $j = 1,2,\ldots p$

Now define

$$\eta_{n+1} = \int_0^\infty f^2(x)\, x^n e^{-Kx} dx$$

$$\xi_{n+1} = \int_0^\infty f(x)\, x^n e^{-Kx} dx \qquad \text{(A.7)}$$

and recall that

$$\int_0^\infty x^n e^{-Kx} dx = \frac{n!}{K^{n+1}}$$

It is now clear why we chose the term e^{-Kx} in the weighting function (A.3). Using the notation of (A.7), (A.6) and (A.5) can be rewritten in the form:

$$\sum_{i=0}^{q} \beta_i \eta_{i+j+1} - \sum_{i=1}^{p} \alpha_i \xi_{i+j+1} = \xi_{j+1} \qquad j = 0,1,\ldots p$$

$$K^{j+1} \sum_{i=0}^{q} \beta_i \xi_{i+j+1} - \sum_{i=1}^{p} \alpha_i \frac{(i+j)!}{K^i} = j! \qquad j = 1,2,\ldots p \qquad \text{(A.8)}$$

By defining

$$T_{ij} = \begin{cases} \dfrac{(i+j-1)!}{K^i} & i = 1,\ldots p \;;\; j = 1,\ldots p \\ K^{j+1}\xi_{i+j+p} & i = p+1,\ldots p+q+1 \;;\; j = 1,\ldots p \\ -\xi_{i+j-p+1} & i = 1,\ldots p \;;\; j = p+1,\ldots p+q+1 \\ \eta_{i+j-2p} & i = p+1,\ldots p+q+1 \;;\; j = p+1,\ldots p+q+1 \end{cases}$$

$$A_j = \begin{cases} (j-1)! & j = 1,\ldots p \\ \xi_{j-p+1} & j = p+1,\ldots p+q+1 \end{cases}$$

$$\theta_j = \begin{cases} \alpha_j & j = 1,\ldots p \\ \beta_{j-p-1} & j = p+1,\ldots p+q+1 \end{cases}$$

we can then write (A.8) in matrix notation as the set of linear equations:

$$\sum_{j=1}^{p+q+1} T_{ij}\theta_j = A_i \quad ; \quad i = 1,\ldots p+q+1 \qquad (A.9)$$

Note that if we had not chosen the weighting function as we did, then we would not have been led to a linear system of equations for the unknowns θ_i, but to a nonlinear system. For numerical work we prefer a linear system to a nonlinear one.

Appendix B

Development of the Minimax Procedure

Before development of the actual procedure for minimax approximation of arbitrary functions with rational polynomials, we first give several theorems and results to place the procedure on a firm mathematical basis.[1]

Let it be required to approximate a given function g, where the real values $y_i = g(t_i)$ at the N distinct points t_i for $i = 0, \ldots N-1$ constitute the only information given concerning g.

Let the set T consist of the N points t_i, $i = 0, \ldots N-1$, where $t_i < t_{i+1}$. The index set S with M elements S_i will be used to indicate a subset of T consisting of the distinct points t_{s_i} where $i = 0, \ldots M-1$ and

$$0 \leq S_i < S_{i+1} \leq N-1$$

thus

$$t_o \leq t_{s_i} < t_{s_{i+1}} \leq t_{N-1}$$

The class F of approximating functions f consists of the functions:

$$f(t) = \frac{\sum_{i=0}^{p} a_i t^i}{\sum_{i=0}^{q} b_i t^i} = \frac{P(t)}{Q(t)}$$

where the numerator degree, p, and the denominator degree, q, are fixed with $a_p \neq 0$ and $b_q \neq 0$.

A function f from class F which gives the best weighted minimax approximation to g over the set T is one for which

$$\max_{t \in T} |f(t) - g(t)|W(t)$$

is minimized. The weight function W(t) is assumed positive for all t_i in T.

---

[1] We follow Kenyon [6] and Golomb [8]

The minimal value will be called λ . That is,

$$\lambda = \min_{f \in F} \max_{t \in T} |f(t) - g(t)|W(t)$$

The deviation at any point t_i is defined as

$$\delta_i = [f(t_i) - g(t_i)]W(t_i)$$

where f may be any member of the class F, not necessarily one giving a best approximation to g.

Theorem: (Existence and Uniqueness) For given $g(t_i)$ and $W(t) > 0$ with $i = 0,1,\ldots N-1$ and fixed p and q, there exists a function f belonging to F such that the maximum deviation

$$\max_i |f(t_i) - g(t_i)|W(t_i)$$

assumes its minimal value λ . The function f is unique provided two functions are considered identical whenever they can be reduced to the same function after cancelling common factors in the numerator and denominator.

Theorem: (Tchebycheff) Let $M = p+q+2 \leq N$ and let f be the best minimax approximation to g on the set T. Then there are at least M points t_{s_i} among the N points t_s of T such that

$$|\delta_{s_i}| = |\delta_{s_{i+1}}| = \lambda \text{ and } \operatorname{sgn} \delta_{s_i} \neq \operatorname{sgn} \delta_{s_{i+1}}$$

where

$$t_{s_i} < t_{s_{i+1}} \text{ and } i = 0,\ldots,M-2$$

The signum of a real number a is defined as $\operatorname{sgn}(a) = 1$ if $a > 0$, $\operatorname{sgn}(a) = 0$ if $a = 0$, and $\operatorname{sgn}(a) = -1$ if $a < 0$.

Theorem: (de la Vallee Poussin) Let $M = p+q+2 \leq N$, with p, q, $g(t_i)$ and $W(t_i)$ given for $i = 0, \ldots M-1$. If f is any member of class F (not necessarily the one yielding the best minimax approximation to $g(t)$) where

$$t_o \leq t_{s_i} < t_{s_{i+1}} \leq t_{N-1}$$

then λ, the absolute value of the maximum weighted deviation for the best approximation, is bounded by the relation

$$\min_i |\delta_{s_i}| \leq \lambda \leq \max_i |\delta_{s_i}|$$

With this background, we now enumerate the steps involved in determining the best approximation function f.

1. From the set T of N distinct points t_i, i = 0 ... N-1, select a subset of M = p + q + 2 points t_{s_i} where M < N and $t_{s_i} < t_{s_{i+1}}$.

2. Find a rational function f(t) = P(t)/Q(t) such that $W_{s_i}[f(t) - y_i] = (-1)^{i+1}\lambda$ is satisfied for minimum $|\lambda|$. This is equivalent to solving the system of equations

$$W_{s_i}\sum_{\sigma=0}^{p} a_\sigma t_{s_i}^\sigma - W_{s_i}Y_{s_i}\sum_{\sigma=0}^{q} b_\sigma t_{s_i}^\sigma = (-1)^{i+1}\lambda\sum_{\sigma=0}^{q} b_\sigma t_{s_i}^\sigma$$

or in matrix terms,

$$Ax = \lambda Bx$$

The column vector x is[1] $[a_o, \ldots, a_p, b_o \ldots b_q]^T$. Matrices A and B are both MxM with elements

$$a_{ij} = \begin{cases} W_{s_i} t_{s_i}^j & 0 \leq j \leq p \\ -W_{s_i}Y_{s_i} t_{s_i}^{j-p-1} & p+1 \leq j \leq M-1 \end{cases}$$

$$b_{ij} = \begin{cases} 0 & 0 \leq j \leq p \\ (-1)^{i+j} t_{s_i}^{j-p-1} & p+1 \leq j \leq M-1 \end{cases}$$

3. Using the rational function f found in step 2, determine a point $t_\xi \in T$ where the weighted deviation

$$\delta(t_i) = W_{s_i}[f(t_i) - g(t_i)]$$

assumes its maximum magnitude $|\delta(t_\xi)|$.

---

[1] A^T denotes the transpose of the matrix A. Thus if $A = [a_{ij}]$ then $A^T = [a_{ji}]$.

4. If $|\delta(t_{\xi})| \leq \lambda$, then the weighted best approximation to $y = g(t)$ on the set T has been formed.

5. If $|\delta(t_{\xi})| \geq \lambda$, then a new subset is selected and the processes are repeated, starting with step 2. The new subset is chosen according to the following rules:

   a. If possible, delete that point t_{s_i} of the old subset which is adjacent to t_{ξ} for which sign $\delta(t_{s_i})$ = sign $\delta(t_{s_{\xi}})$.

   b. Otherwise; namely if $t_{\xi} < t_{s_i}$ or if $t_{\xi} > t_{s_{M-1}}$ and sgn $\delta(t_{s_o})$ or sgn $\delta(t_{s_{M-1}}) \neq$ sgn $\delta(t_{\xi})$ respectively, delate the point of the subset appearing at the opposite end from t_{ξ}, i.e., $t_{s_{M-1}}$ or t_{s_o} respectively. In either case the point t_{ξ} is added to the subset to take the place of the deleted point.

Step 2 involves the solution of nonlinear algebraic equation of the form

$$Ax = \lambda Bx \tag{B.1}$$

where A and B are given MxM matrices. We wish to find the eigenvalue λ, and the eigenvector x.

In general we will find M eigenvectors and M eigenvalues, not necessarily distinct. The eigenvector, x, can be determined only within a multiplicative constant; therefore let us choose $x_k = 1$ as a particular normalization.

The matrix B which arises in the minimax approximation problem has its first $p + 1$ columns all zero. For this reason from this point on, the matrix B will be treated as an M by $q + 1$ matrix where $q + 1 = M - (p + 1)$. We now use the well known power method [7] to solve our reduced system.

Suppose first that the matrix A is singluar. Matrix A is singular if and only if $\lambda = 0$ is an eigenvalue. The corresponding eigenvector is then a solution of the system

$$Ax = 0$$

If $\lambda = 0$ is an eigenvalue, then the rational function approximates the given function with zero error on the set of M points. We then have the usual rational function interpolation problem.

Let us consider now the most prevalent case; namely where A is the non-singular so that $\lambda = 0$ is not an eigenvalue. Letting

$$\mu = \frac{1}{\lambda}$$

and multiplying both sides of equation (B.1) by A^{-1}, then we have

$$Cx = \mu x \tag{B.2}$$

where $C = A^{-1}B$.

Thus we have transformed our problem into solving the system of equations (B.2) for x such that μ is the maximal eigenvalue. The application of the power method can be described easily. A sequence of column vectors $x^{(\nu)}$ starting with $x^{(0)}$ is computed together with a sequence of scalars K_ν. They are defined recursively by

$$Cx^{(\nu)} = y^{(\nu)}$$

$$x^{(\nu+1)} = y^{(\nu)}/K_\nu$$

The scalar K_ν will be taken as the k^{th} component $y_k^{(\nu)}$ of the column vector $y^{(\nu)}$ where k is such that none of the $y_k^{(\nu)}$ are identically zero. The starting vector $x^{(0)}$ can be chosen as any (non-zero) M-dimensional column vector which is not orthogonal to the x, i.e., $(x^{(0)})^T x \neq 0$. Usually one assumes that all the components of $x^{(0)}$ are unity.

If the eigenvalue μ of largest magnitude is real, it can be shown that

$$\mu = \lim_{\nu \to \infty} K_\nu$$

This then completes the method of finding the minimax approximation to a given function.

Appendix C

Frequency Modulation Power Distribution

In section IV, we were lead to a power spectrum of the form

$$P_k(\omega) = A_k \{ \sum_{\ell=1}^{\infty} J_\ell^2(\delta_k) [\delta(\omega-\omega_k - \ell\Omega_k) + \delta(\omega-\omega_k + \ell\Omega_k] + J_o^2(\delta_k)\, \delta(\omega-\omega_k)\}$$

The following table gives the values of $J_n^2(\delta)$ for various δ.

$\delta = 0.10$

n	J_n^2
0	9.9500×10^{-01}
1	2.4937×10^{-03}
2	1.5598×10^{-06}

$\delta = 0.50$

n	J_n^2
0	8.9129×10^{-01}
1	5.9398×10^{-02}
2	9.4784×10^{-04}
3	6.6511×10^{-06}

$\delta = 1.00$

n	J_n^2
0	5.8552×10^{-01}
1	1.9364×10^{-01}
2	1.3202×10^{-02}
3	3.8272×10^{-04}
4	6.1337×10^{-06}

$\delta = 1.50$

n	J_n^2
0	2.6196×10^{-01}
1	3.1129×10^{-01}
2	5.3864×10^{-02}
3	3.7166×10^{-03}
4	1.3848×10^{-04}
5	3.2379×10^{-06}

$\delta = 2.00$

n	J_n^2
0	5.0127×10^{-02}
1	3.3261×10^{-01}
2	1.2449×10^{-01}
3	1.6626×10^{-02}
4	1.1557×10^{-03}
5	4.9556×10^{-05}

$\delta = 2.50$

n	J_n^2
0	2.3409×10^{-03}
1	2.4710×10^{-01}
2	1.9896×10^{-01}
3	4.6915×10^{-02}
4	5.4437×10^{-03}
5	3.8031×10^{-04}
6	1.7847×10^{-05}

$\delta = 3.00$

n	J_n^2
0	6.7627×10^{-02}
1	1.1496×10^{-01}
2	2.3628×10^{-01}
3	9.5519×10^{-02}
4	1.7433×10^{-02}
5	1.8514×10^{-03}
6	1.2882×10^{-04}
7	6.4887×10^{-06}

$\delta = 3.50$

n	J_n^2
0	1.4449×10^{-01}
1	1.8872×10^{-02}
2	2.1034×10^{-01}
3	1.4959×10^{-01}
4	4.1781×10^{-02}
5	6.4709×10^{-03}
6	6.4663×10^{-04}
7	4.5468×10^{-05}

$\delta = 4.00$

n	J_n^2
0	1.4772×10^{-01}
1	4.3617×10^{-03}
2	1.3258×10^{-01}

δ = 4.00 (cont.)

n	J_n^2
3	1.8504×10^{-01}
4	7.9033×10^{-02}
5	1.7446×10^{-02}
6	2.4094×10^{-03}
7	2.3031×10^{-04}
8	1.6230×10^{-05}

δ = 6.00

n	J_n^2
0	2.2693×10^{-02}
1	7.6553×10^{-02}
2	5.8987×10^{-02}
3	1.3171×10^{-02}
4	1.2790×10^{-01}
5	1.3110×10^{-01}
6	6.0435×10^{-02}
7	1.6792×10^{-02}
8	3.1958×10^{-03}
9	4.4797×10^{-04}
10	4.8497×10^{-05}

δ = 8.00

n	J_n^2
0	2.9463×10^{-02}
1	5.5054×10^{-02}
2	1.2767×10^{-02}
3	8.4757×10^{-02}
4	1.1100×10^{-02}
5	3.4512×10^{-02}
6	1.1385×10^{-01}
7	1.0277×10^{-01}
8	4.9932×10^{-02}

δ = 8.00 (cont.)

n	J_n^2
9	1.5956×10^{-02}
10	3.6926×10^{-03}
11	6.5518×10^{-04}
12	9.2617×10^{-05}

δ = 10.00

n	J_n^2
0	6.0484×10^{-02}
1	1.8898×10^{-03}
2	6.4836×10^{-02}
3	3.4081×10^{-03}
4	4.8225×10^{-02}
5	5.4784×10^{-02}
6	2.0905×10^{-04}
7	4.6863×10^{-02}
8	1.0103×10^{-01}
9	8.5179×10^{-02}
10	4.3050×10^{-02}
11	1.5157×10^{-02}
12	4.0157×10^{-03}
13	8.3938×10^{-04}
14	1.4297×10^{-04}
15	2.0321×10^{-05}

δ = 15.00

n	J_n^2
0	2.02335×10^{-04}
1	4.2067×10^{-02}
2	1.7282×10^{-03}
3	3.7643×10^{-02}
4	1.4203×10^{-02}
5	1.7018×10^{-02}

δ = 15.00 (cont.)

n	J_n^2
6	4.2497×10^{-02}
7	1.1877×10^{-03}
8	3.0270×10^{-02}
9	4.8420×10^{-02}
10	8.1129×10^{-03}
11	9.9900×10^{-03}
12	5.6010×10^{-02}
13	7.7681×10^{-02}
14	6.0732×10^{-02}
15	3.2871×10^{-02}
16	1.3486×10^{-02}
17	4.4260×10^{-03}
18	1.1989×10^{-03}
19	2.7468×10^{-04}
20	5.4173×10^{-05}

δ = 20.00

n	J_n^2
0	2.7897×10^{-02}
1	4.4666×10^{-03}
2	2.5709×10^{-02}
3	$9.7814.10^{-03}$
4	1.7074×10^{-02}
5	2.2852×10^{-02}
6	3.0344×10^{-03}
7	3.3937×10^{-02}
8	5.4566×10^{-03}
9	1.5656×10^{-02}
10	3.4775×10^{-02}
11	3.7645×10^{-03}
12	1.4158×10^{-02}
13	4.1675×10^{-02}
14	2.1432×10^{-02}
15	6.5945×10^{-07}
16	2.1077×10^{-02}

δ = 20.00 (cont.)

n	J_n^2
17	5.4334×10^{-02}
18	6.3046×10^{-02}
19	4.7900×10^{-02}
20	2.7141×10^{-02}
21	1.2239×10^{-02}
22	4.5674×10^{-03}
23	1.4477×10^{-03}
24	3.9716×10^{-04}
25	9.5671×10^{-05}

δ = 25.00

n	J_n^2
0	9.2672×10^{-03}
1	1.5712×10^{-02}
2	1.1298×10^{-02}
3	1.1738×10^{-02}
4	1.7502×10^{-02}
5	4.3570×10^{-03}
6	2.5185×10^{-02}
7	1.0339×10^{-04}
8	2.3410×10^{-02}
9	2.1793×10^{-02}
10	5.7520×10^{-03}
11	2.8303×10^{-02}
12	5.3097×10^{-03}
13	9.6595×10^{-03}
14	3.0753×10^{-02}
15	9.5665×10^{-03}
16	3.3305×10^{-03}
17	2.9473×10^{-02}
18	3.0896×10^{-02}
19	6.6313×10^{-03}
20	2.7033×10^{-03}
21	2.7100×10^{-02}
22	5.0433×10^{-02}
23	5.3188×10^{-02}
24	3.9911×10^{-02}
25	2.3393×10^{-02}
26	1.1261×10^{-02}
27	4.5938×10^{-03}
28	1.6225×10^{-03}
29	5.0412×10^{-04}
30	1.3945×10^{-04}
31	3.4678×10^{-05}

Appendix D

Frequency and Amplitude Modulation Parameters

The following tables 1 through 5 give the parameters determined for section VII, part B.

TABLE 1

k	BAND LIMIT	A_k	ω_k	AMPLITUDE MODULATION PARAMETERS			FREQUENCY MODULATION PARAMETERS		
				$\sqrt{A_k}$	$\sqrt{\Delta_k}$	Ω_k	$\sqrt{A_k}$	δ_k	Ω_k
1	4.	618.	2.	24.8	1.19	1.57	24.8	1.00	2.06
2	6.	483.	4.	21.9	5.99	0.51	21.9	1.00	0.49
3	9.	500.	7.	22.3	2.04	0.99	22.3	1.00	1.35
4	11.	355.	9.	18.5	22.68	0.49	18.8	1.00	0.49
5	15.	588.	12.	24.2	1.83	1.43	24.2	1.00	2.59
6	17.	356.	15.	18.8	5.89	0.48	18.8	1.00	0.49
7	20.	498.	18.	22.3	1.93	1.01	22.3	1.00	1.33
8	23.	458.	21.	21.3	1.68	1.02	21.4	1.00	1.23
9	28.	664.	26.	25.7	1.32	1.92	25.7	1.00	3.49
10	31.	397.	29.	19.9	2.03	1.00	19.9	1.00	1.36
11	35.	385.	33.	19.6	1.85	1.45	19.6	1.00	2.67
12	39.	497.	37.	22.2	1.57	1.48	22.3	1.00	2.45
13	43.	517.	40.	22.7	1.39	1.56	22.7	1.00	2.43
14	48.	447.	45.	21.1	1.61	1.87	21.1	1.00	3.96
15	54.	475.	51.	21.7	1.69	2.27	21.8	1.00	6.08
16	60.	526.	56.	22.9	1.60	2.29	22.9	1.00	5.91
17	66.	485.	62.	22.0	1.63	2.23	22.0	1.00	5.67
18	72.	452.	69.	21.2	1.62	2.33	21.2	1.00	6.21
19	80.	478.	75.	21.8	1.70	3.17	21.8	1.00	11.98
20	89.	508.	84.	22.5	1.76	3.50	22.5	1.00	14.95
21	98.	455.	92.	21.3	1.65	3.39	21.3	1.00	13.29
22	111.	463.	104.	21.5	1.51	5.13	21.5	1.00	28.21
23	125.	483.	117.	21.9	1.40	5.47	21.9	1.00	29.71
24	144.	478.	133.	21.8	1.56	7.30	21.8	1.00	58.65
25	170.	476.	156.	21.8	1.56	9.86	21.8	1.00	107.06
26	180.	110.	174.	10.4	2.56	3.22	10.5	3.00	1.77
27	190.	104.	185.	10.1	0.98	4.81	10.2	3.00	1.69
28	201.	109.	195.	10.4	1.13	5.09	10.4	3.00	2.24
29	218.	111.	209.	10.5	1.46	7.20	10.5	3.00	5.97

TABLE 1 (continued)

k	BAND LIMIT	A_k	ω_k	AMPLITUDE MODULATION PARAMETERS			FREQUENCY MODULATION PARAMETERS		
				$\sqrt{A_k}$	$\sqrt{\Delta_k}$	Ω_k	$\sqrt{A_k}$	δ_k	Ω_k
30	235.	100.	225.	9.9	1.96	6.30	10.0	3.00	5.82
31	259.	107.	246.	10.3	1.53	9.10	10.3	3.00	9.98
32	295.	107.	275.	10.3	1.49	14.43	10.3	3.00	24.44
33	354.	105.	321.	10.2	1.47	22.61	10.2	3.00	59.19
34	501.	106.	414.	10.2	1.45	59.20	10.3	3.00	400.27

TABLE 2

k	BAND LIMIT	A_k	ω_k	AMPLITUDE MODULATION PARAMETERS			FREQUENCY MODUATIONS PARAMETERS		
				$\sqrt{A_k}$	$\sqrt{\Delta_k}$	Ω_k	$\sqrt{A_k}$	δ_k	Ω_k
1	4.	618.	2.	24.8	1.19	1.57	24.8	1.00	2.06
2	7.	651.	5.	25.5	1.82	1.01	25.5	1.00	1.29
3	11.	687.	9.	26.2	1.73	1.43	26.2	1.00	2.46
4	15.	588.	12.	24.2	1.83	1.43	24.2	1.00	2.59
5	18.	496.	16.	22.2	1.85	1.00	22.2	1.00	1.28
6	22.	712.	19.	26.6	1.61	1.41	26.6	1.00	2.26
7	27.	574.	24.	23.9	1.65	1.95	23.9	1.00	4.40
8	31.	591.	28.	24.3	1.65	1.46	24.3	1.00	2.47
9	37.	564.	33.	23.7	1.64	2.25	23.7	1.00	5.87
10	41.	642.	39.	25.3	1.60	1.41	25.3	1.00	2.25
11	47.	563.	43.	23.7	1.64	2.25	23.7	1.00	5.86
12	54.	553.	50.	23.5	1.73	2.67	23.5	1.00	8.55
13	61.	605.	57.	25.5	1.69	2.63	24.6	1.00	8.20
14	69.	622.	64.	24.9	1.55	3.04	24.9	1.00	10.18
15	78.	586.	73.	24.2	1.44	3.44	24.2	1.00	12.14
16	89.	636.	83.	25.2	1.69	4.24	25.2	1.00	21.27
17	101.	574.	94.	23.9	1.59	4.74	23.9	1.00	25.20
18	118.	606.	109.	24.6	1.61	6.58	24.6	1.00	49.06
19	139.	595.	127.	24.3	1.54	8.33	24.3	1.00	75.68
20	170.	580.	153.	24.0	1.57	11.88	24.1	1.00	156.35
21	186.	169.	177.	12.8	1.60	6.18	13.0	3.00	4.78
22	201.	154.	193.	12.3	2.66	5.22	12.4	3.00	4.74
23	224.	160.	212.	12.6	1.65	9.12	12.6	3.00	10.71
24	259.	159.	240.	12.5	1.53	13.09	12.6	3.00	20.58
25	320.	160.	286.	12.6	1.58	23.7	12.6	3.00	69.44
26	501.	159.	388.	12.5	1.32	74.66	12.6	3.00	58.09

TABLE 3

k	BAND LIMIT	A_k	ω_k	AMPLITUDE MODULATION PARAMETERS			FREQUENCY MODULATION PARAMETERS		
				$\sqrt{A_k}$	$\sqrt{\Delta_k}$	Ω_k	$\sqrt{A_k}$	δ_i	Ω_k
1	5.	888.	3.	29.7	1.17	1.93	29.8	1.00	3.06
2	10.	893.	7.	29.8	1.78	1.86	29.8	1.00	4.26
3	15.	764.	12.	27.6	1.73	1.86	27.6	1.00	4.16
4	20.	855.	17.	29.2	1.84	1.86	29.2	1.00	4.35
5	26.	748.	22.	27.3	1.72	2.38	27.3	1.00	6.81
6	31.	771.	28.	27.7	1.50	1.87	27.7	1.00	3.71
7	39.	883.	35.	29.7	1.70	3.14	29.7	1.00	11.72
8	45.	706.	41.	26.5	1.64	2.30	26.5	1.00	6.10
9	54.	733.	49.	27.0	1.76	3.47	27.0	1.00	14.68
10	64.	857.	58.	29.2	1.69	3.83	29.2	1.00	17.32
11	74.	753.	68.	27.4	1.66	3.79	27.4	1.00	16.71
12	89.	839.	81.	28.9	1.55	5.83	28.9	1.00	37.21
13	106.	765.	96.	27.6	1.64	6.60	27.6	1.00	50.13
14	130.	789.	117.	28.0	1.48	9.23	28.0	1.00	89.34
15	170.	802.	148.	28.3	1.60	15.56	28.3	1.00	270.04
16	186.	169.	177.	12.9	1.67	6.05	13.0	3.00	4.76
17	201.	154.	193.	12.3	2.66	5.22	12.4	3.00	4.74
18	224.	160.	212.	12.6	1.65	9.12	12.6	3.00	10.71
19	259.	159.	240.	12.5	1.53	13.09	12.6	3.00	20.58
20	320.	160.	286.	12.6	1.58	23.72	12.6	3.00	69.44
21	501.	159.	388.	12.5	1.32	74.66	12.6	3.00	580.09

TABLE 4

k	BAND LIMIT	A_k	ω_k	AMPLITUDE MODULATION PARAMETERS $\sqrt{A_k}$	$\sqrt{\Delta_k}$	Ω_k	FREQUENCY MODULATION PARAMETERS $\sqrt{A_k}$	β_k	Ω_k
1	7.	1269.	3.	35.6	1.23	2.53	25.6	1.00	5.59
2	15.	1276.	10.	35.7	1.56	3.04	35.7	1.00	10.20
3	22.	1208.	18.	34.7	1.66	2.63	34.7	1.00	8.05
4	31.	1166.	26.	34.7	1.30	3.46	34.1	1.00	10.99
5	41.	1207.	36.	34.7	1.52	3.93	34.7	1.00	16.67
6	54.	1117.	47.	33.4	1.64	5.10	33.4	1.00	29.93
7	69.	1227.	61.	35.0	1.60	5.77	35.0	1.00	37.41
8	89.	1222.	78.	34.9	1.62	7.86	34.9	1.00	70.52
9	118.	1181.	102.	34.3	1.64	11.44	34.3	1.00	150.80
10	170.	1176.	140.	34.2	1.55	20.48	34.2	1.00	459.93
11	186.	169.	177.	12.0	1.67	6.05	13.0	3.00	4.76
12	201.	154.	193.	12.3	2.66	5.22	12.4	3.00	4.74
13	224.	160.	212.	12.6	1.65	9.12	12.6	3.00	10.71
14	259.	159.	240.	12.5	1.53	13.09	12.6	3.00	20.58
15	320.	159.	286.	12.6	1.58	23.72	12.6	3.00	69.44
16	501.	159.	388.	12.5	1.32	74.66	12.6	3.00	580.09

TABLE 5

k	BAND LIMIT	A_k	ω_k	AMPLITUDE MODULATION PARAMETERS			FREQUENCY MODULATION PARAMETERS		
				$\sqrt{A_k}$	$\sqrt{\Delta_k}$	Ω_k	$\sqrt{A_k}$	δ_k	Ω_k
1	15.	2546.	7.	50.4	1.49	5.45	50.4	1.00	31.51
2	31.	2375.	22.	48.72	1.71	6.09	48.7	1.00	44.09
3	54.	2324.	41.	48.20	1.38	9.00	48.2	1.00	79.48
4	89.	2450.	69.	49.49	1.50	13.84	49.5	1.00	203.50
5	170.	2357.	121.	48.5	1.38	32.71	48.5	1.00	1046.67
6	194.	247.	181.	15.6	1.63	8.95	15.7	3.00	10.17
7	224.	236.	207.	15.3	1.71	11.81	15.3	3.00	18.48
8	284.	238.	350.	15.4	1.45	23.15	15.4	3.00	61.07
9	501.	240.	358.	15.4	1.17	91.89	15.5	3.00	768.92

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